ELEMENTS

OF THE

INFINITESIMAL CALCULUS

WITH NUMEROUS EXAMPLES AND APPLICATIONS
TO ANALYSIS AND GEOMETRY

BY

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PREFACE.

THE Infinitesimal Calculus is generally considered to be the most difficult branch of pure mathematics to which the attention of the student is directed. It is certainly the most powerful instrument of investigation known to the mathematician, and its philosophy is as profound as its methods are far-reaching and comprehensive. But we believe that its difficulties, in so far as they are not purely algebraic, are due quite as much to the manner in which its first principles are usually exhibited, as to any inherent obscurity in the subject itself.

In the preparation of the following treatise, the attempt has been made to remove all grounds for that feeling of uncertainty which often possesses the student at the very outset, and from which he rarely finds it possible afterward to extricate himself. With this end in view, considerable space has been devoted to an exposition of the doctrine of limits, which has been made the basis of both the Differential and the Integral Calculus.

Many demonstrations might have been abridged, and apparently simplified, by the adoption of the ordinary method of infinitely small quantities; but this would have been, in the opinion of the writer, at the expense of a sound philosophy, for which, in a work intended primarily for educational purposes, the advantage of mere brevity could offer no compensation.

The work is founded mainly upon the excellent philosophical treatise of M. Duhamel, and a large number of the examples have been derived from the works of Hall, Walton, and Todhunter; but many other volumes, whose titles it is needless to mention here, have been consulted, and the writer would hereby acknowledge his indebtedness to all the treatises (American and foreign), relating to the subject, which it has been his privilege to read. He would also tender his obligations to Prof. Schuyler, of Baldwin University, for some valuable criticisms by which he has endeavored to profit; and he is sure that all of his readers will unite with him in this sincere expression of thanks to the publishers for the faultless style in which they have prepared the work for presentation to the public.

LIBERTY, MISSOURI,
May 24th, 1875.

JAMES G. CLARK.

P. S.—Should any who may use the work as a text-book find it too extensive for their purposes, it may be conveniently abridged by omitting the following chapters:

Differential Calculus—VI, IX, X, XI, XIX, XX;
Integral Calculus—IX . . . . XVI, inclusive.

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THE DIFFERENTIAL CALCULUS.

CHAPTER I.

THE METHOD OF LIMITS.

SECTION 1.—Definitions and Fundamental Propositions.

1. QUANTITY, when made the subject of mathematical investigation, is to be considered under one of two aspects; viz., as **constant** or as **variable.**

A constant quantity is one whose value remains fixed after having once been assigned; while a variable quantity is one whose value is either continually changing, or may be supposed capable of such change. Thus, the distance between two fixed points, measured on the straight line joining them, is constant; the distance from the center of an ellipse to its circumference is variable.

2. Whatever may be the law according to which a quantity varies, it will usually be found that there is some one value toward which the variable may be made to approach indefinitely, without ever reaching it; and it will sometimes occur that there are two such values within which the variable is confined.

Thus, in the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots\ldots,$$

the terms themselves are evidently converging toward *zero,* while the sum of the terms approaches *two* indefinitely.

The distance between two fixed points being *unity*, the distance from either of them to a variable intermediate point must vary between *zero* and *unity*. This distance can never be either zero or unity; for in either case the third point would coincide with one of the fixed points, and thereby violate the hypothesis that it is between them.

3. The extreme value toward which the value of a variable may be made to approach, or converge, indefinitely, is called the **limit** of the variable.

The *inferior* limit of a variable is the value toward which it converges in decreasing.

The *superior* limit of a variable is the value toward which it converges in increasing.

It is evident that *zero* and *infinity* are the inferior and superior limits of positive numbers.

4. An **infinitesimal** is a variable which has *zero* for its limit. For example, the difference between the value of any variable and its limit is an infinitesimal; since, as the variable tends toward its limit, this difference becomes less and less indefinitely, without ever reaching zero.

It is evident from this definition that however small may be any given value of an infinitesimal, it will, in tending toward its limit, zero, become smaller still, without ever being actually reduced to zero. We may, therefore, say that *an infinitesimal is a variable which may assume a value less than that of any assignable finite magnitude, however small*; and when we attribute such a value to an infinitesimal, it becomes what is usually called an **infinitely small quantity.** The theory of the Calculus does not, in general, require us *actually to attribute* infinitely small values to these variables, and therefore we have preferred to use the term infinitesimal as descriptive of the class rather than the term infinitely small, which is of more limited application.

5. **Proposition.**—*If two variables are equal in every stage of their variation, and each tends toward a limit, their limits are equal.*

For, since the variables are equal, they may be substituted each for the other. Performing this substitution, we shall have the first tending toward the limit of the second, and *vice versa;* and since it is obviously impossible for a variable to tend toward two different limits at the same time, it follows that the two limits must be equal.

Corollary 1.—It is evident that the limits of the two members of an equation which involve variables are equal.

Corollary 2.—A constant quantity may be called its own limit, and hence if one member of an equation is constant, the value of that constant is the limit of the other member.

6. Proposition.—*The limit to the algebraic sum of any number of variables is equal to the algebraic sum of their limits.*

Let x, y, z . . . be the variables, and designate their limits by lim. (x), lim. (y), etc. Let α, β, γ, etc., be the differences between these variables and their respective limits.

Then we shall have

$$x = \text{lim. } (x) + \alpha.$$
$$y = \text{lim. } (y) + \beta.$$
$$z = \text{lim. } (z) + \gamma.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x + y + z = \text{lim. } (x) + \text{lim. } (y) + \text{lim. } (z) \ldots + \alpha + \beta + \gamma \ldots \quad \text{(a).}$$

Now, the limits of α, β, γ . . . being zero [Art. 4], the limit to their sum is evidently zero. The limit of the second member of (a) is, therefore, lim. (x) + lim. (y) + lim. (z); and since the limits of the two members are the same, we shall have

$$\text{lim. } (x + y + z) = \text{lim. } (x) + \text{lim. } (y) + \text{lim. } (z).$$

7. Proposition.—*The limit to the product of two or more variables is equal to the product of their limits.*

Employing the same symbols as before, we have

$$xyz = \text{lim. } (x).\ \text{lim. } (y).\ \text{lim. } (z) + \text{terms involving } \alpha, \beta, \gamma \text{ as factors.}$$

The limit to each of these last terms being evidently zero, we have, by taking the limit of each side of the equation,

$$\lim. (xyz) = \lim. (x). \lim. (y). \lim. (z).$$

Corollary 1.—If the product of several variables is constant, the product of their limits is also constant, and equal to that of the variables themselves.

$$\text{Let } xyz = c.$$
$$\text{Then } \lim. (xyz) = \lim. (c) = c.$$

Corollary 2.—If two variables are reciprocal, their limits are also reciprocal.

$$\text{Let } xy = 1, \text{ or } x = \frac{1}{y}.$$
$$\text{Then } \lim. (xy) = \lim. (x). \lim. (y) = \lim. (1) = 1.$$
$$\therefore \lim. (x) = \frac{1}{\lim. (y)}.$$

8. Proposition.—*The limit to the quotient of two variables is equal to the quotient of their limits.*

Since $\frac{x}{y} = x\left(\frac{1}{y}\right)$, we have

$$\lim. \left(\frac{x}{y}\right) = \lim. (x). \lim. \left(\frac{1}{y}\right). \tag{1}$$

Put $\frac{1}{y} = z$; then $yz = 1$, and $\lim. (y). \lim. (z) = 1$.

$$\therefore \lim. (z) = \lim. \left(\frac{1}{y}\right) = \frac{1}{\lim. (y)};$$

and by substitution in (1),

$$\lim. \left(\frac{x}{y}\right) = \lim. (x). \lim. \left(\frac{1}{y}\right) = \lim. (x). \frac{1}{\lim. (y)} = \frac{\lim. (x)}{\lim. (y)}.$$

9. Proposition.—*The limit to the* n^{th} *power of a variable is equal to the* n^{th} *power of its limit.*

1st. When n is an integer, we have, by Art. 7,

$$\lim. (x^n) = \lim. (xxx \ldots x) = \lim. (x). \lim. (x) \ldots \lim. (x)$$
$$= \{\lim. (x)\}^n.$$

2d. Let n be any fraction, as $\frac{p}{q}$; whence $x^n = x^{\frac{p}{q}}$.

Put $x^{\frac{1}{q}} = y$; whence $x = y^q$, and $x^{\frac{p}{q}} = y^p$. Then we have

$$\text{lim. } (x) = \text{lim. } (y^q) = \{\text{lim. } (y)\}^q.$$

Consequently,

$$\left\{\text{lim. } (x)\right\}^{\frac{1}{q}} = \text{lim. } (y), \text{ and}$$

$$\left\{\text{lim. } (x)\right\}^{\frac{p}{q}} = \{\text{lim. } (y)\}^p = \text{lim. } (y^p) = \text{lim. } \left(x^{\frac{p}{q}}\right).$$

SECTION II.—Magnitudes Considered as Limits.

10. Since any quantity may be considered as the limit to a variable, and since any number of variables may have the same limit, it is evident that we may regard any given quantity as the limit to a variable of simpler form than itself. Whenever, therefore, we wish to determine a relation between two quantities, we may often facilitate the operation by considering them as the limits of simpler variables, then finding the relation between these new variables, and finally, by means of the propositions established in the preceding section, passing to the limits, the relation between which will be the required relation between the given quantities.

11. Magnitudes may be considered as the limits of variables from several different points of view.

1st. We may consider any number as the limit to the sum of the terms of a converging series of finite quantities.

EXAMPLE.—Unity is the limit to the sum of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

2d. A quantity may be considered as the limit to the sum of an infinite series of infinitesimals.

EXAMPLE.—The length of a given straight line is equal to the sum of the lengths of the parts into which it may be divided. The greater the number of parts, the less is the length of each. If, then, the number of parts is variable, the length of each part is evidently a variable which has *zero* for its limit, while the number of parts has *infinity* for its limit. The length of the line is evidently equal to the limit to the sum of its infinitesimal parts.

3d. We may regard any quantity as the *limit to the ratio of two variables.*

Ex.—$\frac{1}{2}$ is the limit to the ratio of the two variables

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots, \text{ and}$$
$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots$$

12. Proposition.—*The limit to the sum of an indefinite number of positive infinitesimals is not changed when we replace them by others whose ratios to them have unity for their limit.*

Let $a, b, c, \ldots l$ be infinitesimals whose sum tends toward a fixed limit as their number is increased.

Let $a', b', c', \ldots l'$ be other infinitesimals, such that each of the ratios $\frac{a}{a'}$, $\frac{b}{b'}$, etc., shall have unity for its limit.

Now, it is a principle of Algebra, that if we have a series of fractions with positive denominators, the ratio of the sum of the numerators to the sum of the denominators will be intermediate in value between the greatest and least of the fractions. Hence,

$$\frac{a+b+c+\ldots l}{a'+b'+c'+\ldots l'}$$

is comprised between the greatest and least of the fractions $\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}$, etc. If, therefore, the limit to each of these fractions is unity, it follows that

$$\lim.\left\{\frac{a+b+c+\ldots l}{a'+b'+c'+\ldots l'}\right\}=1; \text{ or}$$

$$\lim.\ (a+b+c+\ldots l)=\lim.\ (a'+b'+c'+\ldots l').$$

Corollary 1.—If either of these limits be equal to a constant, the other will be equal to the same constant.

Corollary 2.—If the limit to the ratio of the corresponding elements of the two sums be l, the limit to the ratio of the two sums themselves will evidently be l; and this principle enables us, under certain circumstances, to reduce the investigation of the *limits of sums* to that of the *limits of ratios*, which, as we shall see in the sequel, is a much simpler process than the former.

13. Proposition.—*The limit to the ratio of two variables is not changed when we replace them by others whose ratios to them have unity for their limit.*

Let a and b be two variables. Let a' and b' be two others, such that $\lim. \frac{a}{a'} = 1$, and $\lim. \frac{b}{b'} = 1$; whence

$$\lim. \frac{a'}{a} = 1, \text{ and } \lim. \frac{b'}{b} = 1.$$

Now, since $\frac{a}{b} = \frac{a'}{b'} \cdot \frac{b'}{b} \cdot \frac{a}{a'}$, we have

$$\lim. \frac{a}{b} = \lim. \frac{a'}{b'} \times \lim. \frac{b'}{b} \times \lim. \frac{a}{a'}.$$

$$\text{But } \lim. \frac{b'}{b} = 1, \text{ and } \lim. \frac{a}{a'} = 1.$$

$$\therefore \lim. \frac{a}{b} = \lim. \frac{a'}{b'}.$$

Corollary.—This proposition is evidently true, according to Art. 12, if for the word *ratio* we substitute *sum.*

14. Proposition.—*When the limit to the ratio of two variables is unity, their difference is an infinitesimal, and conversely.*

Let a and a' be two variables and δ their difference. Then $a' = a + \delta$, and dividing by a', $1 = \frac{a}{a'} + \frac{\delta}{a'}$.

Hence, if the limit to $\frac{a}{a'}$ is unity, the limit to $\frac{\delta}{a'}$ must be *zero*, and δ must be an infinitesimal.

Conversely, if the limit to $\frac{\delta}{a'}$ is zero, or if δ is an infinitesimal, the limit to $\frac{a}{a'}$ must be *unity.*

Note.—In the preceding demonstration a and a' are assumed to be variables whose limits are different from zero.

Corollary 1.—*The limit to the ratio of two variables is not changed when we replace them by others from which they differ by infinitesimals.*

This follows directly from the preceding propositions, but its great importance warrants us in giving the following independent demonstration, for which we are indebted to Professor Schuyler.

Let a and b be two variables, the limit to whose ratio is r; and let x and y be two infinitesimals.

$$\text{Then lim. } \frac{a}{b} = r, \text{ and lim. } \frac{b}{a} = \frac{1}{r}.$$

$$\text{Also lim. } \frac{a}{a+x} = 1, \text{ and lim. } \frac{b}{b+y} = 1.$$

$$\therefore \text{ lim. } \left\{ \frac{b}{a} \times \frac{a+x}{b+y} \right\} = 1; \ \therefore \text{ lim. } \frac{b}{a} \times \text{lim. } \frac{a+x}{b+y} = 1;$$

$$\therefore \frac{1}{r} \times \text{lim. } \frac{a+x}{b+y} = 1; \ \therefore \text{ lim. } \frac{a+x}{b+y} = r;$$

$$\therefore \text{ lim. } \frac{a}{b} = \text{lim. } \frac{a+x}{b+y}.$$

Corollary 2.—The limit to the sum of any number of variables is not changed when we replace them by others from which they differ by infinitesimals. This is evident, since the infinitesimals disappear in taking the limits.

15. Of Different Orders of Infinitesimals.—We have seen that the limit to the ratio of two variables may be any finite number or zero, and it may also be itself a variable, either infinitesimal or otherwise. Thus, let R designate the radius vector of an ellipse, let i be an infinitesimal increment to the radius, and let A be the semi-major axis. Then

$$\text{lim. } \frac{R+i}{A} = \frac{R}{A};$$

and since R is itself variable, while A is constant, $\frac{R}{A}$ is a variable quantity.

The statement made above is true whether the variables have finite limits or whether they are infinitesimals, and this circumstance gives rise to infinitesimals of different orders.

Let x be any variable whose limit is different from *zero*, and let a, b, c, d, etc., be infinitesimals; let the limit to the ratio of a and x be zero; then we shall call a an infinitesimal of the *first* order; and any infinitesimal whose ratio to a has a *finite* limit, will also be an infinitesimal of the first order. It is evident that *an infinitesimal of the first order is simply an infinitesimal part of a variable whose limit is finite.*

Similarly, *an infinitesimal of the second order is an infinitesimal part of one of the first order;* one of the third order is an infinitesimal part of one of the second order; and so on.

15′. Proposition.—*The product of two infinitesimals of the first order is of the second order.*

Let x be an infinitesimal of the second order, and let y, z be two of the first order; then we may evidently, in accordance with the preceding definitions, write

$$\frac{x}{y} = z; \text{ whence } x = yz.$$

15″. Proposition.—*To determine algebraic expressions for the infinitesimals of different orders.*

Let $a, a_1, a_2, a_3 \ . \ . \ . \ a_n$ be infinitesimals arranged in order, a being the one with which the others are to be compared. Designate $\lim \frac{a_1}{a}$ by k, a finite quantity. Then we shall have $\frac{a_1}{a} = k + \delta$, in which δ is an infinitesimal which disappears at the limit. Whence

$$a_1 = a\ (k + \delta)$$

which will be the general expression for an infinitesimal of the first order. Also,

$$\frac{a_2}{a} = a\ (k + \delta), \text{ or } a_2 = a^2\ (k + \delta) \qquad (2);$$

$$\frac{a_3}{a} = a^2\ (k + \delta), \text{ or } a_3 = a^3\ (k + \delta) \qquad (3);$$

and, generally,

$$a_n = a^n\ (k + \delta) \tag{n}$$

for the infinitesimal of the n^{th} order.

Corollary to (15), (15′), (15″).—The corollaries to Prop. 14 are true of infinitesimals as well as of finite variables. For, let a and b be two infinitesimals, and let x and y be infinitesimals of higher order than a and b. Then, word for word, as in Cor. 1., Prop. 14, we may prove that

$$\text{lim. } \frac{a}{b} = \text{lim. } \frac{a+x}{b+y}.$$

Corollary 2, Prop. 14, may be extended to the case of infinitesimals as follows:

Since $\text{lim. } \dfrac{a}{a+x} = 1$, we have $\text{lim. } (a) = \text{lim. } (a + x)$;

and since $\text{lim. } \dfrac{b}{b+y} = 1$, we have $\text{lim. } (b) = \text{lim. } (b + y)$.

$$\therefore \text{ lim. } (a + b) = \text{lim. } (a) + \text{lim. } (b) = \text{lim. } (a + x)$$
$$+ \text{ lim. } (b + y) = \text{lim. } (a + x + b + y).$$

Therefore, in general, *the limit to the sum or ratio of two infinitesimals is not changed when they are replaced by others from which they differ by infinitesimals of higher order than themselves.*

SECTION III.—Increments and Derivatives.

16. When two variables are in any way dependent upon each other, any change attributed to either of them will usually effect a change in the value of the other. Such changes are called **Increments.**

17. Let y be any function of x, designated by $F(x)$, and let h be an infinitesimal increment of the first order assigned to x, converting y into $y + k$. Then we shall have

$$y = F(x);$$
$$y + k = F(x + h);$$
$$k = F(x + h) - F(x);$$
$$\frac{k}{h} = \frac{F(x+h) - F(x)}{h};$$

and

$$\lim. \frac{k}{h} = \lim. \left\{ \frac{F(x+h) - F(x)}{h} \right\}.$$

The limit to the ratio of k and h is called the **derivative** of y with respect to x, and is designated by $F'(x)$. We therefore have

$$F'(x) = \lim. \frac{k}{h} = \lim. \left\{ \frac{F(x+h) - F(x)}{h} \right\}.$$

Now, since [Art. 4] the difference between a variable and its limit is an infinitesimal, we shall have (calling this difference δ)

$$\frac{k}{h} = F'(x) + \delta, \text{ and}$$

$$k = h\ F'(x) + \delta h.$$

Again, since δ and h are infinitesimals of the first order, the product [Art. 15'] is an infinitesimal of the second order, and *we may therefore substitute* h F'(x) *for* k *whenever the latter appears as one of the terms of a ratio or series whose limits we wish to find.*

18. The derivative of a function will enable us to determine the manner in which the function varies between any assumed values.

Thus we have seen that

$$k = h\ F'(x) + \delta h = h\ (F'(x) + \delta).$$

Now, if for any given value of x, $F'(x)$ is not *zero*, δ being an infinitesimal will, as it tends toward its limit *zero*, necessarily become less than $F'(x)$, and the sign of the factor $F'(x) + \delta$ will depend upon that of $F'(x)$. Therefore, supposing h to be positive, k will, when δ has reached a value less than $F'(x)$, have the same sign as $F'(x)$.

Consequently, $F(x)$ is *increasing* for all values of x which render $F'(x)$ positive; and $F(x)$ is *decreasing* for all values of x which render $F'(x)$ *negative.*

The reverse will evidently be true if we consider h as negative.

Corollary.—It is obvious that, of two functions of x, that changes the more rapidly whose derivative for any given value of x is the greater.

19. Proposition.—*Let* $y = F(x)$ *be a function of* x, *subject to the following conditions:*

1st. That the increment k of y produced by the increment h of x shall tend toward *zero* as h tends toward *zero*.

2d. That, if in passing from the value x_0 to X, the value of x varies continually in the same sense, then y will also vary continually in the same sense in passing from y_0 to Y.

Then we say that *for every value of* x *between* x_0 *and* X, *the derivative* $F'(x)$, *or lim.* $\frac{k}{h}$, *has a determinate value which is, in general, finite.*

For, if we divide the interval $X - x_0$ into n equal parts, and designate by k_1, k_2, etc., the corresponding increments of y, we shall have

$$Y - y_0 = k_1 + k_2 + \ldots k_n; \quad X - x_0 = nh;$$

and therefore,

$$\frac{Y - y_0}{X - x_0} = \frac{k_1 + k_2 + \ldots k_n}{nh} = \frac{\frac{k_1}{h} + \frac{k_2}{h} + \ldots \frac{k_n}{h}}{n}.$$

That is, the constant ratio $\frac{Y - y_0}{X - x_0}$ is the arithmetical mean of the ratios $\frac{k_1}{h}$, $\frac{k_2}{h}$, etc.

Let n be indefinitely increased. The terms of each of these ratios will then tend to *zero*, and the limit to each ratio will be a value of lim. $\frac{k}{h}$, or of the derivative $F'(x)$. $\frac{Y - y_0}{X - x_0}$ will then be the mean of the derivatives of $F(x)$ taken for all possible values of x between x_0 and X; and since the value of this mean, *in general*, is finite, it follows that the derivatives themselves must be finite.

This does not, however, preclude the possibility of there being, within the prescribed limits x_0 and X, particular values of x for which the value of the derivative may be either *zero* or *infinity*.

20. *If for all values of* x *between* x_0 *and* X *the value of the derivative be zero, it is clear that all the values of* y *between* y_0 *and* Y *will be equal to each other, and therefore independent of* x. For, in that case, we shall have

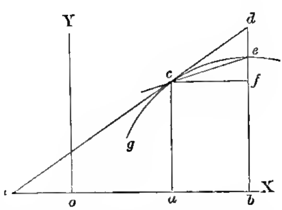
$$\frac{Y-y_0}{X-x_0}=0; \text{ therefore, } Y-y_0=0, \text{ and } Y=y_0;$$

and the same will be true if for Y we substitute any intermediate value of y.

GEOMETRICAL ILLUSTRATIONS.

21.—1st. Let it be required to draw a tangent to the curve *gce* at the point *c*, and let $y=F(x)$ be the equation of that curve, referred to the rectangular axes *o*X, *o*Y. Designate the co-ordinates of *c* by x and y, and those of *e* by $x+h$, $y+k$. Draw the secant line *ce*, and the tangent *cd*. If, now, we suppose the point *e* to move toward *c*, the line *ce* will tend to coincide with *cd*, the position of which will evidently be the *limiting* position of *ce*, which the latter is always approaching, but which it can never reach without ceasing to be a secant, and thereby violating the original hypothesis.

Fig. 1



In this movement of the point *e* the lines *df* and *ef* evidently tend toward *zero*, as also does *cf*; moreover, *df* and *ef*, as they approach *c*, tend toward equality with each other; *df*, *ef*, and *cf* are, therefore, infinitesimals, and the limit to the ratio of *df* and *ef* is unity.

The position of the tangent *cd* will be determined if we

can find the tangent of the angle dcf which it makes with the axis of x. The value of this tangent is (if we consider the trigonometrical radius as unity) $\frac{df}{cf}$; and it might seem impossible to determine this ratio without first finding the value of df, which evidently depends on the position of the tangent line itself, the very thing for which we are searching. But, since the limit to the ratio of df and ef is *unity*, we are [Art. 14, Cor. 1] at liberty to substitute ef for df, and the determination of the ratio $\frac{df}{cf}$ is thus reduced to that of the limit to the ratio $\frac{ef}{cf}$.

We therefore have

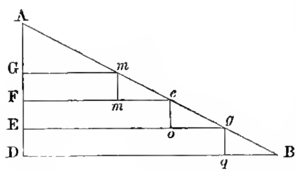
$$\text{tang } dcf = \frac{df}{cf} = \lim. \frac{ef}{cf} = \lim. \frac{k}{h} = F'(x).$$

Hence, *the tangent of the angle which the tangent line to a curve makes with the axis of abscissas*—the axes being rectangular—*is equal to the derivative of the ordinate of the point of tangency.*

2d. Let it be required to find the area of a right-angled triangle, ABD.

Designate the base BD by b, and the altitude AD by a. Divide the altitude into any number n of equal parts, ED, FE, etc., and complete the rectangles gD, oF, etc. Now, each rectangle, as gD, will differ from the corresponding trapezoid BE by an elementary triangle, Bgq, and the sum of all the rectangles will differ from the given triangle by the sum of these elementary triangles.

Fig. 2



But, if n be increased indefinitely, the area of each of these triangles will decrease, and the limit to each triangle is evidently *zero*. The given triangle is obviously equal to the limit to the sum of the rectangles and the sum of the

triangles. But the limit to the sum of the triangles is *zero*, since the limit to each is *zero*. Hence, the given triangle is the limit to the sum of the rectangles.

$$\text{Now, let } Bq = go = em, \text{ etc.} = h;$$

$$\text{also, let ED} = \text{FE, etc.} = \frac{a}{n}.$$

Then we shall have

$$\text{rectangle } q\text{E} = (b - h)\frac{a}{n};\ o\text{F} = (b - 2h)\frac{a}{n};\ m\text{G} = (b - 3h)\frac{a}{n},\ \text{etc.}$$

Hence,

$$\text{ABD} = \text{lim.}\ \left\{(b - h)\frac{a}{n} + (b - 2h)\frac{a}{n} + \ldots (b - nh)\frac{a}{n}\right\}$$

$$= \text{lim.}\ \left\{nb\,\frac{a}{n} - \frac{ah}{n}(1 + 2 + \ldots n)\right\}$$

$$= \text{lim.}\ \left\{ab - \frac{ah}{n}\cdot\frac{n^2 + n}{2}\right\} = \text{lim.}\ \left\{ab - \frac{ahn}{2} - \frac{ah}{2}\right\}.$$

But, whatever may be the number of rectangles, we have always $nh = b$, and the limit of h is *zero*. Hence, since the limit to the sum is equal to the sum of the limits,

$$\text{ABD} = \text{lim.}\ (ab) - \text{lim.}\left(\frac{ab}{2}\right) - 0 = ab - \frac{ab}{2} = \frac{ab}{2}.$$

CHAPTER II.

FUNDAMENTAL PRINCIPLES OF THE DIFFERENTIAL CALCULUS.

SECTION I.—Definitions.

22. When any relation exists between two or more variables by means of which their values may be determined, they are said to be **functions** of each other.

In such cases it is usual, as a matter of convenience, to *select* one or more of the variables in terms of which to express the others. The variables so selected are called **independent**, and the others **dependent** variables, or simply **functions.**

23. Functions are either **explicit** or **implicit.**

An explicit function is one whose value may be determined by performing the operations indicated.

An implicit function is one whose evaluation requires the solution of an equation.

In the equation $y = a + bx + cx^2$, y is an explicit function of x.

In the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, y is an implicit function of x.

24. Functions are either **algebraic** or **transcendental.** An algebraic function is one whose relation to the independent variable is, or may be, expressed in a finite number of algebraic terms. The examples in Art. 23 are algebraic functions. Every function which can not be so expressed is a transcendental function.

$$y = \log\ x;\ y = a^x;\ y = \sin\ x;\ y = \sin^{-1} x,$$

are examples of transcendental functions.

25. When we wish to indicate, in a general way, that y is an explicit function of x, we make use of some such expression as

$$y = F(x);\ y = f(x);\ y = \phi(x),$$

the characters F, f, ϕ designating different functions. These expressions are read simply, y *equal to the* F *function of* x, etc.

When we wish to indicate that y is an implicit function of x, we write

$$F(x.\ y) = 0;\ f(x.\ y) = 0;\ \phi(x.\ y) = 0,$$

and read, *the* F *function of* x *and* y *equal to* zero, etc.

26. When a function depends for its value *directly* upon one or more independent variables, it is called a **simple function.**

Thus, $y = 4x + 3z$ is a simple function of x and z.

When several functions are accumulated upon each other, the first is said to be a **function of functions.**

$$\text{Thus, if } y = F(z)\,;\;\; z = f(u)\,;\;\; u = \phi(x),$$

y is a function of functions.

When a variable is a function of several variables, each of which is a function of the same variable, the first is said to be a **compound function** of the last.

$$\text{Thus, if } y = F(z.\ u.\ t)\,;\;\; z = f(x)\,;\;\; u = \phi(x)\,;\;\; t = \psi(x),$$

y is a compound function of x.

If one variable is a function of another, the second is an **inverse function** of the first.

27. A **variable** is **continuous** when, in passing from one value to another, it passes successively through all intermediate values. When this condition is not fulfilled, the variable is **discontinuous.**

A **function** is **continuous** when, in making the variable on which it depends vary continuously, it is constantly *real*, and also varies continuously. A function may be continuous for all values of the independent variable within certain limits, and discontinuous beyond those limits.

28. We have seen in the first section of Chapter I., that when two variables are always equal their limits are also equal, and that in order to find the limit to any algebraic combination of variables, it is sufficient to replace each variable by its limit. Hence, if

$$F(x.\ y.\ z\ .\ .\ .) = f(x.\ y.\ z\ .\ .\ .),$$

we shall also have

$$\lim.\ F(x.\ y.\ z\ .\ .\ .) = \lim.\ f(x.\ y.\ z\ .\ .\ .).$$

If, now, $a.\ b.\ c\ .\ .\ .$ are the limits of $x.\ y.\ z\ .\ .\ .$, then

$$\lim.\ F(x.\ y.\ z\ .\ .\ .) = F(a.\ b.\ c\ .\ .\ .),$$
$$\text{and } \lim.\ f(x.\ y.\ z\ .\ .\ .) = f(a.\ b.\ c\ .\ .\ .).$$

Hence, finally,

$$F(a.\ b.\ c\ .\ .\ .) = f(a.\ b.\ c\ .\ .\ .).$$

If, therefore, we wish to discover the relation which exists between several variables, it is sufficient to consider them as the limits to other variables which may be of simpler form or of a character more readily dealt with; to determine the relation between these new variables; and, finally, in the algebraic expression of this relation, to replace them by their limits.

29. It has been demonstrated in Art. 19, that the ratio of the two infinitesimal increments of a function and its variable has, in general, a determinate finite limit, which we have called the *derivative of the function* with respect to the variable.

It is the primary **object of the Differential Calculus** *to investigate and establish methods of determining the derivatives of functions under all possible forms and combinations.*

SECTION II.—Derivatives and Differentials.

30. The infinitesimal increment of a variable or function is called its **difference,** and is designated by the symbol Δ. Thus, Δx is the difference of x.

The derivative of a function with respect to a given variable is designated by the same symbol which denotes the function, affected with an accent. Thus, if we designate a function by $y = F(x)$, then $F'(x)$ is the derivative of y with respect to x; and since the derivative of a function is the limit to the ratio of the infinitesimal increments of the func-

tion and the independent variable, we shall have, in accordance with our notation,

$$F'(x) = \lim. \frac{\Delta y}{\Delta x} \qquad (1),$$

in which Δy and Δx are infinitesimals.

31. If, now, we represent by α a certain quantity which tends toward *zero* at the same time with Δx, we shall have, in accordance with the theory of limits,

$$\frac{\Delta y}{\Delta x} = F'(x) + \alpha, \text{ or } \Delta y = \Delta x.\ F'(x) + \alpha \Delta x,$$

in which $\alpha \Delta x$ is an infinitesimal of higher order than Δx; and therefore [Art. 17], *whenever* Δy *enters into the terms of a ratio or series whose limits we wish to find, we may replace it by* Δx. F′(x). Consequently, in all such cases, and *in such cases only,* we may write, as rigorously exact, the equation

$$\Delta y = \Delta x.\ F'(x) \qquad (2),$$

$$\text{or} \qquad \frac{\Delta y}{\Delta x} = F'(x) \qquad (3).$$

32. Equations (2) and (3) being rigorously exact only when they are limiting equations, it is found expedient in practice to substitute for Δy and Δx the symbols dy and dx, which are used *only* in limiting equations or expressions, *and represent quantities whose ratio is equal to the limit to the ratio of* Δy *and* Δx.

Instead, then, of the above equations, we may write the following:

$$\frac{dy}{dx} = F'(x) \qquad (4),$$

$$\text{and} \qquad dy = F'(x).\ dx \qquad (5),$$

which are rigorously exact.

The expressions dy and dx are called the **differentials** of y

and x; and $\frac{dy}{dx}$, being the quantity by which it is necessary to multiply dx in order to obtain dy, is called the **differential co-efficient** of y with respect to x. It represents symbolically the derivative of y with respect to x, together with the algebraic process by which it is obtained.

A comparison of equations (2) and (5) will show that dy and dx may be considered as infinitesimals whose ratios to Δy and Δx, respectively, have the limit *unity*, and which may be of simpler form than these last, or *vice versa*; and, if x be taken as the independent variable, dx may evidently be considered as equal to Δx.

33. If we have in the same investigation several functions (y, z, u, etc.) of the same variable x, and represent by Δy, Δz, etc., their differences with respect to Δx, and by dy, dz, etc., their corresponding differentials, we shall have

$$\frac{\Delta y}{\Delta z} = \frac{\Delta y \,.\, \Delta x}{\Delta x \,.\, \Delta z} = \frac{\frac{\Delta y}{\Delta x}}{\frac{\Delta z}{\Delta x}};$$

and, passing to the limits,

$$\lim. \frac{\Delta y}{\Delta z} = \frac{\lim. \frac{\Delta y}{\Delta x}}{\lim. \frac{\Delta z}{\Delta x}}, \quad \text{or,}$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}.$$

Hence, the derivative of one variable with respect to another is equal to the ratio of their derivatives taken with respect to any third variable of which they are both functions.

SECTION III.—Differentiation in General.

34. *The operation of finding the derivative and differential of a function is called* **differentiation.**

All functions, whatever may be their form, are either simple functions or made up of simple functions, and we proceed to show how the differentiation of any function may be made to depend upon that of the simple functions of which it is composed.

[1] DIFFERENTIATION OF SIMPLE FUNCTIONS.

Let $y = F(x)$ be any simple function of x.

Then we shall have, as has previously been shown,

$$\Delta y = F(x + \Delta x) - F(x);$$

$$\frac{\Delta y}{\Delta x} = \frac{F(x + \Delta x) - F(x)}{\Delta x};$$

$$\lim. \frac{\Delta y}{\Delta x} = \lim. \frac{F(x + \Delta x) - F(x)}{\Delta x};$$

$$\text{or,} \quad \frac{dy}{dx} = F'(x), \text{ and } dy = F'(x).\ dx.$$

Hence, to differentiate a simple function of a variable, *give to the variable an increment, and find the corresponding increment of the function. The limit to the ratio of these increments will be the derivative of the function. Place this derivative equal to the differential co-efficient, multiply by the differential of the variable, and the result will be the differential of the function.*

Scholium.—If we suppose the independent variable to vary uniformly, so that Δx is the same for all values of x, dx may be taken as the *rate of variation* of x, and dy, being the infinitesimal change in y due to the corresponding change in x, may be considered as the rate of variation of y.

[2] DIFFERENTIATION OF A FUNCTION OF FUNCTIONS.

Let u be a function of x, determined by the series of operations

$$u = F(z)\,;\ z = f(y)\,;\ y = \phi(x)\,;$$

and let it be required to find the derivative of u with respect to x.

Let Δu, Δz, Δy be the differences of u, z, y. Then we shall have

$$\frac{\Delta u}{\Delta x} = \frac{\Delta u}{\Delta z} \cdot \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x},$$

and, passing to the limits,

$$\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx} = F'(z) \cdot f'(y) \cdot \phi'(x).$$

Hence, *the derivative of a function* u *of* x, *through several intermediate functions, is equal to the product of the derivatives of the whole series of functions.*

[3] DIFFERENTIATION OF INVERSE FUNCTIONS.

Let $y = F(x)$, and $x = f(y)$.

Since these two equations are the same under different forms, they will give the same increments for the variables. We shall therefore have

$$F'(x) = \lim.\ \frac{\Delta y}{\Delta x} = \frac{1}{\lim.\ \frac{\Delta x}{\Delta y}} = \frac{1}{f'(y)} = \frac{1}{f'(F(x))}.$$

The equation

$$F'(x) = \frac{1}{f'(F(x))}$$

expresses the method of performing this differentiation.

[4] DIFFERENTIATION OF COMPOUND FUNCTIONS.

In order to determine the derivative of a compound function we shall first find an expression for the infinitesimal increment of a function of several variables.

$$\text{Let } y = F(u.\ v),$$

and let Δy, Δu, Δv be their increments.

Then we shall have

$$\Delta y = F(u + \Delta u,\ v + \Delta v) - F(u.\ v),$$

which can be written

$$\Delta y = F(u + \Delta u,\ v) - F(u.\ v) + F(u + \Delta u,\ v + \Delta v) - F(u + \Delta u,\ v),$$

or

$$\Delta y = \frac{F(u + \Delta u,\ v) - F(u.\ v)}{\Delta u}\,\Delta u. + \frac{F(u + \Delta u,\ v + \Delta v) - F(u + \Delta u,\ v)}{\Delta v}\,\Delta v.$$

In this equation the co-efficient of Δu differs by an infinitesimal from the derivative of $F(u.\ v)$ with respect to u, v being regarded as constant; and in accordance with the theory of limits this co-efficient may be replaced by that derivative, $\frac{dy}{du}$.

In like manner, the co-efficient of Δv differs by an infinitesimal from the derivative of $F(u + \Delta u,\ v)$ with respect to v, u being regarded as constant; and this latter derivative differs by an infinitesimal from the derivative of $F(u.\ v)$ with respect to v. Hence we may substitute $\frac{dy}{dv}$ for the derivative of $F(u + \Delta u,\ v)$, and this for the co-efficient of Δv.

We therefore have

$$\Delta y = \frac{dy}{du}.\ \Delta u + \frac{dy}{dv}.\ \Delta v.$$

It must be observed, however, that since quantities can be substituted for others from which they differ by infinitesi-

mals only in cases where *limits* are to be taken, we must, in the general case, add to the above value of Δy an infinitesimal α, which shall disappear when we pass to the limit.

We thus have

$$\Delta y = \frac{dy}{du}. \Delta u + \frac{dy}{dv}. \Delta v + \alpha,$$

an equation which is rigorously correct, and is independent of any relation between u and v.

If now we have

$$y = F(u. v. z \ldots),$$

where u, v, z . . . are functions of x, we shall have

$$\Delta y = \frac{dy}{du}. \Delta u + \frac{dy}{dv}. \Delta v + \frac{dy}{dz}. \Delta z + \ldots + \alpha.$$

Dividing by Δx, and passing to the limits, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}. \frac{du}{dx} + \frac{dy}{dv}. \frac{dv}{dx} + \frac{dy}{dz}. \frac{dz}{dx} \ldots \qquad \text{(A)},$$

or, multiplying by dx,

$$dy = \frac{dy}{du}. du + \frac{dy}{dv}. dv + \frac{dy}{dz}. dz + \ldots \qquad \text{(B)}.$$

These two equations, (A) *and* (B), *are formulas for obtaining the derivative and differential of a compound function of a variable,* and are much more readily remembered than the rules which might be derived from them.

NOTE.—In these equations $\frac{dy}{du}$, $\frac{dy}{dv}$, etc., are the *partial* derivatives of y with respect to each of the variables, and $\frac{dy}{dx}$ in the first member of (A) is the *total* derivative of y with respect to the variable x, on which all the others depend. If any one of the variables, as u, should be equal to x, then we would have

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dx} = \frac{dy}{dx},$$

a partial derivative of the same form with the total derivative in the first member. In like manner, dy in the first member of (B) is the *total* differential of y, while dy in the second member is a partial differential.

Particular attention is necessary in using these expressions to prevent the confusion of total *with* partial *differentials and derivatives.* The difficulty is sometimes obviated by inclosing the total derivatives or differentials in brackets.

[5] DIFFERENTIATION OF IMPLICIT FUNCTIONS.

(a). Let $F(x.\ y) = 0$ be an equation expressing that y is an implicit function of x.

Taking the derivatives according to formula (A) in the last case, we have

$$\frac{dF(x.\ y)}{dx} + \frac{dF(x.\ y)}{dy} \cdot \frac{dy}{dx} = 0.$$

Whence

$$\frac{dy}{dx} = -\frac{\frac{dF(x.\ y)}{dx}}{\frac{dF(x.\ y)}{dy}}, \text{ and } dy = -\frac{\frac{dF(x.\ y)}{dx}}{\frac{dF(x.\ y)}{dy}} \cdot dx.$$

These expressions may be somewhat abbreviated by substituting u for $F(x.\ y)$.

(b). If we have $m - 1$ equations with m variables, the differentials of these variables may be determined in terms of the variables and the differential of any one of them.

Let

$$F(x.\ y.\ z\ .\ .\ .) = 0;\ f(x.\ y.\ z\ .\ .\ .) = 0;\ \phi(x.\ y.\ z\ .\ .\ .) = 0,$$

etc., be the equations.

Differentiating with respect to x, y, z . . ., and representing the first members of the equations by F, f, ϕ, etc., we have

$$\frac{dF}{dx}.\ dx + \frac{dF}{dy}.\ dy + \frac{dF}{dz}.\ dz = 0,$$

$$\frac{df}{dx}.\ dx + \frac{df}{dy}.\ dy + \frac{df}{dz}.\ dz = 0,$$

$$\frac{d\phi}{dx}.\ dx + \frac{d\phi}{dy}.\ dy + \frac{d\phi}{dz}.\ dz = 0,$$

. . . .

From these $m - 1$ equations the values of dy, dz, etc., can be determined in terms of x, y, z . . . and dx.

CHAPTER III.

DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE.

35. Problem 1.—*To differentiate* $y = x \pm a$.

We have

$$y + \Delta y = x + \Delta x \pm a;$$

$$\text{whence} \quad \Delta y = \Delta x, \text{ and } \frac{\Delta y}{\Delta x} = 1$$

$$\therefore \frac{dy}{dx} = \text{lim. } \frac{\Delta y}{\Delta x} = 1; \text{ and } dy = dx.$$

Therefore, *if a constant be connected with a variable by the sign + or —, it disappears in differentiation.*

Note.—In this and the succeeding problems it is to be understood that y is a function of x.

Problem 2.—*To differentiate* $y = ax$.

We have

$$y + \Delta y = a(x + \Delta x) = ax + a\Delta x,$$

$$\text{whence} \quad \Delta y = a\Delta x, \text{ and } \frac{\Delta y}{\Delta x} = a.$$

$$\therefore \frac{dy}{dx} = \lim. \frac{\Delta y}{\Delta x} = a, \text{ and } dy = adx.$$

Therefore, *if the variable be multiplied by a constant, the differential contains that constant as a factor.*

Problem 3.—*To differentiate any power of a variable.*

Let $y = x^m$.

Then we shall have

$$y + \Delta y = (x + \Delta x)^m = x^m + mx^{m-1}\Delta x + \frac{m(m-1)}{1.\ 2}x^{m-2}(\Delta x)^2 + \text{etc.}$$

Whence, by subtraction,

$$\Delta y = mx^{m-1}\Delta x + \frac{m(m-1)}{1.\ 2}x^{m-2}(\Delta x)^2 + \text{etc.};$$

and, dividing by Δx,

$$\frac{\Delta y}{\Delta x} = mx^{m-1} + \frac{m(m-1)}{1.\ 2}x^{m-2}\Delta x + \text{etc.}$$

$$\therefore \lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = mx^{m-1}, \text{ and } dy = mx^{m-1}dx.$$

Therefore, *the derivative of any power of a variable is found by diminishing the exponent by unity and multiplying the result by the original exponent; and the differential is found by multiplying this derivative by the differential of the variable.*

Note.—Since the binomial formula is true for all values of m, this rule for differentiating is correct for all powers.

Problem 4.—*To differentiate* $y = \sqrt{x}$.

We have $x = y^2$; whence $dx = 2ydy$ (by Prob. 3).

$$\therefore\ dy = \frac{dx}{2y} = \frac{dx}{2\sqrt{x}}.$$

Therefore, *to differentiate a radical of the second degree, differentiate the variable under the radical sign and divide the result by twice the radical.*

Problem 5.—*To differentiate*

$$y = as + bz + cu + kv + \text{etc.},$$

where y, s, z, etc., are functions of x.

Let the increment Δx be given to x; then y, s, z, etc., will be converted into $y + \Delta y$, $s + \Delta s$, $z + \Delta z$, etc., and we shall have

$$y + \Delta y = a(s + \Delta s) + b(z + \Delta z) + c(u + \Delta u) + k(v + \Delta v) + \text{etc.}$$

$$\therefore\ \Delta y = a\Delta s + b\Delta z + c\Delta u + k\Delta v + \text{etc.},$$

$$\frac{\Delta y}{\Delta x} = a\frac{\Delta s}{\Delta x} + b\frac{\Delta z}{\Delta x} + c\frac{\Delta u}{\Delta x} + k\frac{\Delta v}{\Delta x} + \text{etc.},$$

and

$$\lim.\ \frac{\Delta y}{\Delta x} = a.\ \lim.\ \frac{\Delta s}{\Delta x} + b.\ \lim.\ \frac{\Delta z}{\Delta x} + c.\ \lim.\ \frac{\Delta u}{\Delta x} + k.\ \lim.\ \frac{\Delta v}{\Delta x} + \text{etc.},$$

$$\text{or}\quad \frac{dy}{dx} = a\frac{ds}{dx} + b\frac{dz}{dx} + c\frac{du}{dx} + k\frac{dv}{dx} + \text{etc.}$$

$$\therefore\ dy = ads + bdz + cdu + kdv + \text{etc.}$$

Therefore, *the differential of the algebraic sum of any number of functions of the same variable is found by taking the algebraic sum of their differentials.*

Problem 6.—*To differentiate the product of any number of functions of the same variable.*

(*a*). Let us take $y = uz$.

Then we shall have

$$y + \Delta y = (u + \Delta u).\ (z + \Delta z) = uz + u\Delta z + z\Delta u + \Delta u.\ \Delta z,$$
$$= y + u\Delta z + z\Delta u + \Delta u.\ \Delta z.$$

$$\therefore\ \Delta y = u\Delta z + z\Delta u + \Delta u.\ \Delta z;\ \text{and}$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta z}{\Delta x} + z\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}.\ \Delta z.$$

Passing to the limits, and observing that, since Δz enters as a factor into the last term, the limit of this term is *zero*, we have

$$\frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx},\ \text{and}$$

$$dy = udz + zdu.$$

(*b*). Let the function be $y = uzv$.

Designating the product of u and z by s, we have

$$y = sv,$$

and, as in the previous case,

$$dy = sdv + vds. \tag{1}$$

But, since $s = uz$, we have

$$ds = udz + zdu.$$

$\therefore$ by substitution in (1),

$$dy = uzdv + uvdz + vzdu.$$

The same process can evidently be extended to any number of functions, and we shall have the following general rule:

To differentiate the product of any number of functions of the same variable, multiply the differential of each function by the product of all the other functions, and take the algebraic sum of the products so obtained.

Corollary.—Dividing the equation

$$dy = uvdz + uzdv + vzdu$$

by $y = uvz$, we obtain

$$\frac{dy}{y} = \frac{dz}{z} + \frac{dv}{v} + \frac{du}{u}.$$

Wherefore, *if the differential of each function be divided by the function itself, the sum of the quotients will be equal to the differential of the product of the functions divided by the product.*

Problem 7.—*To differentiate the quotient of two functions of a variable.*

Let the expression be $y = \frac{u}{v}$.

Then we shall have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}; \text{ whence}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v} \qquad (1).$$

Dividing by Δx, and passing to the limits, observing that the limit to the denominator in the second member of (1) is v^2, we have

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \text{ and } dy = \frac{vdu - udv}{v^2}.$$

Therefore, *to differentiate a fraction, multiply the denominator by the differential of the numerator, the numerator by the differential of the denominator, subtract the second product from the first, and divide by the square of the denominator.*

Corollary 1.—If the denominator be constant, then the second term in the differential vanishes, and we have

$$dy = \frac{vdu}{v^2} = \frac{du}{v}.$$

Corollary 2.—If the numerator be constant, then the first term in the differential vanishes, and we have

$$dy = \frac{-udv}{v^2}.$$

Problem 8.—*To differentiate* $y = \log(x)$.

We have

$$y + \Delta y = \log(x + \Delta x) = \log\left\{x(1 + \frac{\Delta x}{x})\right\}$$
$$= \log x + \log\left(1 + \frac{\Delta x}{x}\right).$$

$$\therefore \quad \Delta y = \log\left(1 + \frac{\Delta x}{x}\right) = M\left(\frac{\Delta x}{x} - \frac{(\Delta x)^2}{2x^2}, \text{ etc.}\right);$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = M\left(\frac{1}{x} - \frac{\Delta x}{2x^2}, \text{ etc.}\right);$$

$\therefore$ passing to the limits,

$$\frac{dy}{dx} = M\left(\frac{1}{x}\right), \text{ and } dy = \frac{Mdx}{x}.$$

Therefore, *to differentiate the logarithm of a variable, divide the differential of the variable by the variable and multiply by the modulus of the system.*

Corollary.—If the logarithm be taken in the Naperian system, the modulus is unity, and we have

$$dy = \frac{dx}{x}.$$

We shall in all cases, unless it is otherwise stated, understand our logarithms to be referred to the Naperian system.

Problem 9.—*To differentiate* $y = a^x$

Passing to logarithms, we have

$$\log y = x \log a.$$

Therefore, by differentiation,

$$\frac{dy}{y} = dx \log a; \ dy = y \ dx \log a; \text{ or}$$

$$da^x = a^x \ dx \log a.$$

Corollary.—If a is equal to e, the Naperian base, then $\log a = \log e = 1$, and therefore,

$$d\, e^x = e^x \ dx.$$

Problem 10.—*To differentiate* $y = \sin x$.

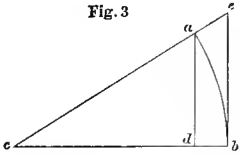
In order to solve this problem it is necessary to demonstrate in the first place that the limit to the ratio of the sine and tangent is *unity*. For this purpose it is sufficient to observe that,

$$\text{since} \quad \text{tang} = \frac{\sin}{\cos}, \quad \text{we have} \quad \frac{\sin}{\text{tang}} = \cos,$$

and therefore,

$$\lim.\left(\frac{\sin}{\text{tang}}\right) = \lim.\ (\cos) = 1.$$

Fig. 3



We may also notice that, since the length of the arc of a circle is intermediate in value between the sine and tangent, we shall have

$$\lim.\left(\frac{\sin}{\text{tang}}\right) = \lim.\left(\frac{\text{arc}}{\text{tang}}\right) = \lim.\left(\frac{\text{arc}}{\sin}\right) = \lim.\left(\frac{\sin}{\text{arc}}\right) = 1.$$

Now, resuming our equation $y = \sin x$, and giving to x an increment, Δx, we shall have

$$y + \Delta y = \sin\,(x + \Delta x) = \sin x.\ \cos \Delta x + \cos x.\ \sin \Delta x;$$

$$\therefore\ \Delta y = \sin x.\ \cos \Delta x + \cos x.\ \sin \Delta x - \sin x;$$

$$\frac{\Delta y}{\Delta x}=\frac{\sin x}{\Delta x}.\cos \Delta x+\cos x.\frac{\sin \Delta x}{\Delta x}-\frac{\sin x}{\Delta x};$$

and

$$\lim.\frac{\Delta y}{\Delta x}=\lim.\left(\frac{\sin x}{\Delta x}\right).\lim.(\cos \Delta x)$$

$$+\lim.\left(\frac{\sin \Delta x}{\Delta x}\right).\lim.(\cos x)-\lim.\left(\frac{\sin x}{\Delta x}\right).$$

But

$$\lim.(\cos \Delta x)=1,\text{ and }\lim.\left(\frac{\sin \Delta x}{\Delta x}\right)=1.$$

$$\therefore\ \lim.\frac{\Delta y}{\Delta x}=\lim.\left(\frac{\sin x}{\Delta x}\right)+\lim.(\cos x)-\lim.\left(\frac{\sin x}{\Delta x}\right);$$

or

$$\frac{dy}{dx}=\cos x,\text{ and }dy=\cos x\ dx=d\sin x.$$

Problem 11.—*To differentiate* y = *cos* x.

We have

$$y=\cos x=\sin(90^\circ-x).$$

$$\therefore\ \text{(by Prob. 10)},\ dy=\cos(90^\circ-x)\ d(90^\circ-x)$$

$$=\sin x\ d(-x)=-\sin x\ dx=d\cos x.$$

Problem 12.—*To differentiate* y = *tang* x.

We have

$$y=\frac{\sin x}{\cos x}.$$

$$\therefore\ \text{(by Prob. 7)},\ dy=\frac{\cos x\ d\sin x-\sin x\ d\cos x}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}.dx=\frac{1}{\cos^2 x}\ dx=\sec^2 x\ dx=d\operatorname{tang} x.$$

Problem 13.—*To differentiate* y = *cot* x.

We have

$$y = \text{tang}(90° - x).$$

$$\therefore\ dy = \sec^2(90° - x)\ d(90° - x)$$
$$= -\text{cosec}^2\ x\ dx = d\ \cot\ x.$$

Problem 14.—*To differentiate* y = *sec* x.

We have

$$y = \sec\ x = \frac{1}{\cos\ x}.$$

$$\therefore\ dy = d\left(\frac{1}{\cos\ x}\right) = \frac{\sin\ x\ dx}{\cos^2\ x} = \frac{\sin\ x}{\cos\ x}.\ \frac{1}{\cos\ x}.\ dx$$
$$= \text{tang}\ x.\ \sec\ x\ dx = d\ \sec\ x.$$

Problem 15.—*To differentiate* y = *cosec* x.

We have

$$y = \text{cosec}\ x = \sec(90° - x).$$

$$\therefore\ dy = d\ \sec(90° - x) = \text{tang}(90° - x).\ \sec(90° - x)\ d(90° - x)$$
$$= -\cot\ x.\ \text{cosec}\ x\ dx = d\ \text{cosec}\ x.$$

Problem 16.—*To differentiate* y = *versin* x.

We have

$$y = \text{versin}\ x = 1 - \cos\ x.$$

$$\therefore\ dy = d(1 - \cos\ x) = \sin\ x\ dx = d\ \text{versin}\ x.$$

Problem 17.—*To differentiate* y = *coversin* x.

We have

$$y = \text{coversin}\ x = \text{versin}(90° - x).$$

$$\therefore\ dy = d\ \text{versin}(90° - x) = \sin(90° - x)\ d(90° - x)$$
$$= -\cos\ x\ dx = d\ \text{coversin}\ x.$$

Problem 18.—*To differentiate* $y = \sin^{-1} x$.
We have

$$x = \sin y.$$

$$\therefore\ dx = \cos y\, dy = \sqrt{1 - \sin^2 y}\, dy = \sqrt{1 - x^2}\, dy;$$

$$\text{whence,}\quad dy = \frac{dx}{\sqrt{1 - x^2}} = d \sin^{-1} x.$$

Problem 19.—*To differentiate* $y = \cos^{-1} x$.
We have

$$x = \cos y.$$

$$\therefore\ dx = -\sin y\, dy = -\sqrt{1 - \cos^2 y}\, dy = -\sqrt{1 - x^2}\, dy;$$

$$\text{whence,}\quad dy = -\frac{dx}{\sqrt{1 - x^2}} = d \cos^{-1} x.$$

Problem 20.—*To differentiate* $y = \text{tang}^{-1} x$.
We have

$$x = \text{tang}\ y.$$

$$\therefore\ dx = \sec^2 y\, dy = (1 + \text{tang}^2 y)\, dy = (1 + x^2)\, dy;$$

$$\text{whence,}\quad dy = \frac{dx}{1 + x^2} = d\ \text{tang}^{-1} x.$$

Problem 21.—*To differentiate* $y = \cot^{-1} x$.
We have

$$x = \cot y$$

$$\therefore\ dx = -\text{cosec}^2 y\, dy = -(1 + \cot^2 y)\, dy = -(1 + x^2)\, dy;$$

$$\text{whence,}\quad dy = -\frac{dx}{1 + x^2} = d \cot^{-1} x.$$

Problem 22.—*To differentiate* $y = \sec^{-1} x$.
We have

$$x = \sec y.$$

$$\therefore\ dx = \text{tang } y.\ \sec y\, dy = \sqrt{\sec^2 y - 1}.\ \sec\ y\, dy$$
$$= x\sqrt{x^2-1}\, dy;$$

$$\text{whence,} \quad dy = \frac{dx}{x\sqrt{x^2-1}} = d \sec^{-1} x.$$

Problem 23.—*To differentiate* y = cosec^{-1} x.

We have

$$x = \text{cosec } y.$$

$$\therefore\ dx = -\cot y.\ \text{cosec } y\, dy = -\sqrt{\text{cosec}^2 y - 1}.\ \text{cosec } y\, dy$$
$$= -x\sqrt{x^2-1}\, dy;$$

$$\text{whence,} \quad dy = -\frac{dx}{x\sqrt{x^2-1}} = d\ \text{cosec}^{-1} x.$$

Problem 24.—*To differentiate* y = *versin*$^{-1}$ x.

We have

$$x = \text{versin } y.$$

$$\therefore\ dx = \sin y\, dy = \sqrt{2 \text{ versin } y - (\text{versin } y)^2}\, dy$$
$$= \sqrt{2x - x^2}\, dy;$$

$$\text{whence,} \quad dy = \frac{dx}{\sqrt{2x-x^2}} = d \text{ versin}^{-1} x.$$

Problem 25.—*To differentiate* y = *coversin*$^{-1}$ x.

We have

$$x = \text{coversin } y.$$

$$\therefore\ dx = -\cos y\, dy = -\sqrt{2 \text{ coversin } y - (\text{coversin } y)^2}\, dy$$
$$= -\sqrt{2x - x^2}.\ dy;$$

$$\text{whence,} \quad dy = -\frac{dx}{\sqrt{2x-x^2}} = d \text{ coversin}^{-1} x.$$

36. The preceding problems embrace the methods of differentiating all the known *simple* functions of a single variable. Problems 1 to 7, inclusive, embrace all the **algebraic** functions; 8 and 9 are the **logarithmic,** and its inverse, the **exponential,** functions; 10 to 25 are the direct and inverse **circular** or **trigonometric** functions, in each of which the radius is taken as *unity.*

The methods of differentiating all simple functions being known, the differentiation of functions of functions, compound functions, and implicit functions, can be readily effected by means of the formulas already provided for such cases.

37. EXAMPLES.

1. Differentiate $y = 3x + 7$.

According to Problem 1, the term 7 will disappear in differentiation, and by Problem 2, we shall have

$$\frac{dy}{dx} = 3, \text{ or } dy = 3dx.$$

2. Differentiate $y = x^5$.

According to Problem 3, we have

$$\frac{dy}{dx} = 5x^4. \quad \therefore \ dy = 5x^4\,dx.$$

3. Differentiate $y = 3\sqrt{2x}$.

We have, by Problems 2 and 4,

$$dy = \frac{3d(2x)}{2\sqrt{2x}} = \frac{6dx}{2\sqrt{2x}} = \frac{3dx}{\sqrt{2x}}.$$

4. Differentiate $y = 5x^4 + 7x^3 - 4x^2 + 6x - 5$.

We have, by Problem 5,

$$dy = 5d(x^4) + 7d(x^3) - 4d(x^2) + 6dx;$$

$\therefore$ differentiating each term,

$$dy = 5(4x^3dx) + 7(3x^2dx) - 4(2xdx) + 6dx$$
$$= 20x^3dx + 21x^2dx - 8xdx + 6dx.$$

5. Differentiate $y = (4x^5)(3x^2)$.

We have, by Problem 6,

$$dy = 4x^5d(3x^2) + 3x^2d(4x^5)$$
$$= 4x^5(6xdx) + 3x^2(20x^4dx)$$
$$= 24x^6dx + 60x^6dx = 84x^6dx.$$

6. Differentiate $y = ax^3(bx^2 + cx + h)$.

We have

$$dy = 3ax^2dx(bx^2 + cx + h) + ax^3(2bxdx + cdx)$$
$$= 3abx^4dx + 3acx^3dx + 3ahx^2dx + 2abx^4dx + acx^3dx$$
$$= 5abx^4dx + 4acx^3dx + 3ahx^2dx.$$

The same result would be obtained by performing the indicated multiplication and then differentiating.

7. Differentiate $y = 48x^9$.

By Problem 3, we have,

$$dy = 432x^8dx.$$

Again, resolving $48x^9$ into three factors, we have

$$y = 2x^2.\ 4x^3.\ 6x^4.$$

Differentiating this by Problem 6 (*b*), we have

$$dy = 24x^7.\ 4xdx + 12x^6.\ 12x^2dx + 8x^5.\ 24x^3dx$$
$$= (96x^8 + 144x^8 + 192x^8)dx = 432x^8dx.$$

8. Differentiate $y = \dfrac{3x^2}{6x}$.

We have, by Problem 7,

$$dy = \frac{6xd(3x^2) - 3x^2d(6x)}{36x^2} = \frac{36x^2dx - 18x^2dx}{36x^2} = \tfrac{1}{2}dx.$$

9. Differentiate $y = \dfrac{x^5}{a}$.

Since a is constant, we have

$$dy = \frac{5x^4dx}{a}.$$

10. Differentiate $y = \dfrac{a}{x^5}$.

We have, by Problem 7, Cor. 2,

$$dy = \frac{-5ax^4dx}{x^{10}} = \frac{-5adx}{x^6}.$$

11. Differentiate $y = \log(3x^4)$.

We have, by Problem 8,

$$dy = \frac{d(3x^4)}{3x^4} = \frac{12x^3dx}{3x^4} = \frac{4dx}{x}.$$

12. Differentiate $y = a^{3x}$.

We have, by Problem 9,

$$dy = a^{3x} \log a \; d(3x) = 3a^{3x} \log a \; dx.$$

13. Differentiate $y = a^{x^2}$.

We have

$$dy = a^{x^2} \log a \; d(x^2)$$
$$= 2xa^{x^2} \log a \; dx.$$

14. Differentiate $y = e^{ax}$.

We have

$$dy = e^{ax}d(ax) = ae^{ax}dx.$$

15. Differentiate $y = \sin(x^2)$.

We have, by Problem 10,

$$dy = \cos(x^2)d(x^2) = 2x\cos(x)^2 dx.$$

16. Differentiate $y = \cos(5x)$.

We have, by Problem 11,

$$dy = -\sin(5x)d(5x) = -5\sin(5x)\,dx.$$

17. Differentiate $y = \text{tang}(ax)$.

We have, by Problem 12,

$$dy = \sec^2(ax)d(ax) = a\sec^2(ax)\,dx.$$

18. Differentiate $y = \sec(ax)$.

We have, by Problem 14,

$$dy = \text{tang}(ax).\sec(ax)d(ax) = a\,\text{tang}(ax).\sec(ax)dx.$$

19. Differentiate $y = \text{versin}(ax)$.

We have, by Problem 16,

$$dy = \sin(ax)d(ax) = a\sin(ax)dx.$$

20. Differentiate $y = a\sin^{-1}\frac{x}{a}$.

We have, by Problem 18,

$$dy = \frac{a\frac{dx}{a}}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{adx}{\sqrt{a^2-x^2}}.$$

21. Differentiate $y = a\cos^{-1}\frac{x}{a}$.

We have, by Problem 19,

$$dy = \frac{-a\frac{dx}{a}}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{-adx}{\sqrt{a^2-x^2}}.$$

22. Differentiate $y = a \text{ tang}^{-1} \dfrac{x}{a}$.

We have, by Problem 20,

$$dy = \frac{a\dfrac{dx}{a}}{1 + \dfrac{x^2}{a^2}} = \frac{a^2 dx}{a^2 + x^2}.$$

23. Differentiate $y = a \cot^{-1} \dfrac{x}{a}$.

We have, by Problem 21,

$$dy = -\frac{a\dfrac{dx}{a}}{1 + \dfrac{x^2}{a^2}} = \frac{-a^2 dx}{a^2 + x^2}.$$

24. Differentiate $y = a \sec^{-1} \dfrac{x}{a}$.

We have, by Problem 22,

$$dy = \frac{a\dfrac{dx}{a}}{\dfrac{x}{a}\sqrt{\dfrac{x^2}{a^2} - 1}} = \frac{a^2 dx}{x\sqrt{x^2 - a^2}}.$$

25. Differentiate $y = a \text{ cosec}^{-1} \dfrac{x}{a}$.

We have, by Problem 23,

$$dy = \frac{-a\dfrac{dx}{a}}{\dfrac{x}{a}\sqrt{\dfrac{x^2}{a^2} - 1}} = \frac{-a^2 dx}{x\sqrt{x^2 - a^2}}.$$

26. Differentiate $y = a \text{ versin}^{-1} \dfrac{x}{a}$.

We have, by Problem 24,

$$dy = \frac{a\dfrac{dx}{a}}{\sqrt{2\dfrac{x}{a} - \dfrac{x^2}{a^2}}} = \frac{a dx}{\sqrt{2ax - x^2}}.$$

27. Differentiate $y = a \text{ coversin}^{-1} \dfrac{x}{a}$.

We have, by Problem 25,

$$dy = -\frac{a\dfrac{dx}{a}}{\sqrt{2\dfrac{x}{a} - \dfrac{x^2}{a^2}}} = \frac{-a\,dx}{\sqrt{2ax - x^2}}.$$

37'. The preceding examples have been given simply for the purpose of illustration. The following are intended as examples for practice; they are given without regard to order, and it will be noticed that some of them are quite complicated.

1. Differentiate $y = (a + bx^m)^n$.

Assume $a + bx^m = z$. $\therefore$ $y = z^n$, $dy = nz^{n-1}\,dz$, and $dz = mbx^{m-1}\,dx$.

$\therefore$ by substitution,

$$dy = bmn(a + bx^m)^{n-1}x^{m-1}\,dx.$$

We may also differentiate this example by the rule for powers and obtain the same result.

2. Differentiate $y = x(1 + x^2)(1 + x)^2$.

We have

$$dy = (1 + x^2)(1 + x)^2dx + x(1 + x)^2 2x dx + x(1 + x^2)2(1 + x)dx = ?$$

3. Differentiate $y = \sqrt{ax + bx^2}$.

We have

$$dy = \frac{a + 2bx}{2\sqrt{ax + bx^2}}\,dx.$$

4. Differentiate $y = \dfrac{5x^4}{(x^2 + 3)^3}$.

We have

$$dy = \frac{20x^3(x^2 + 3)^3 - 3(x^2 + 3)^2\,2x.\,5x^4}{(x^2 + 3)^6}\,dx = ?$$

5. Differentiate $y = \log(a + bx + cx^2 + hx^3)$.

$$\text{Ans. } \frac{dy}{dx} = \frac{b + 2cx + 3hx^2}{a + bx + cx^2 + hx^3}.$$

6. Differentiate $y = \sqrt{x + \sqrt{1 + x^2}} = \{x + (1 + x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$.
We have

$$dy = \tfrac{1}{2}\{x + (1 + x^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} d\{x + (1 + x^2)^{\frac{1}{2}}\}$$

$$= \tfrac{1}{2}\{x + (1 + x^2)^{\frac{1}{2}}\}^{-\frac{1}{2}}.\{1 + \tfrac{1}{2}(1 + x^2)^{-\frac{1}{2}} 2x\}dx$$

$$= \frac{1}{2} \frac{1 + \dfrac{x}{\sqrt{1 + x^2}}}{\sqrt{x + \sqrt{1 + x^2}}} dx = \frac{1}{2} \frac{\sqrt{x + \sqrt{1 + x^2}}}{\sqrt{1 + x^2}} dx.$$

7. Differentiate $y = \sqrt[4]{x}.\ \sqrt[3]{\sqrt{x} + 1} = x^{\frac{1}{4}}(x^{\frac{1}{2}} + 1)^{\frac{1}{3}}$.
We have

$$dy = \tfrac{1}{4}x^{-\frac{3}{4}}(x^{\frac{1}{2}} + 1)^{\frac{1}{3}}dx + \tfrac{1}{3}x^{\frac{1}{4}}(x^{\frac{1}{2}} + 1)^{-\frac{2}{3}}.\ \tfrac{1}{2}x^{-\frac{1}{2}}dx$$

$$= \frac{(x^{\frac{1}{2}} + 1)^{\frac{1}{3}}dx}{4x^{\frac{3}{4}}} + \frac{dx}{6x^{\frac{1}{4}}(x^{\frac{1}{2}} + 1)^{\frac{2}{3}}}$$

$$= \frac{5\sqrt{x} + 3}{12\sqrt[4]{x^3}\sqrt[3]{(\sqrt{x} + 1)^2}} dx.$$

8. Differentiate $y = \dfrac{x}{x - \sqrt{1 - x^2}}$.

9. Differentiate $y = \dfrac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}$.

Multiplying both terms of the second member by the numerator, we have

$$y = \frac{(\sqrt{1 + x} + \sqrt{1 - x})^2}{2x} = \frac{1 + \sqrt{1 - x^2}}{x}.$$

$$\frac{dy}{dx} = ?$$

10. Differentiate $y = \sqrt{x + \sqrt{x + \sqrt{x + \text{etc.},}}}$ continued indefinitely.

Squaring this equation, we obtain

$$y^2 = x + y, \text{ or } y^2 - y = x.$$

$$\therefore\ 2y\,dy - dy = dx, \text{ and } dy = \frac{dx}{2y - 1}.$$

11. Differentiate $y = \log\,(x + \sqrt{1 + x^2})$.

We have

$$dy = \frac{d\{x + \sqrt{1 + x^2}\}}{x + \sqrt{1 + x^2}} = \frac{1 + x(1 + x^2)^{-\frac{1}{2}}}{x + \sqrt{1 + x^2}}\,dx = \frac{dx}{\sqrt{1 + x^2}}.$$

12. Differentiate $y = x(a^2 + x^2)\sqrt[3]{a^2 - x^2}$.

Taking the logarithm of each side of this equation, we have

$$\log y = \log x + \log(a^2 + x^2) + \tfrac{1}{3}\log(a^2 - x^2).$$

$$\therefore\ \frac{dy}{y} = \frac{dx}{x} + \frac{d(a^2 + x^2)}{a^2 + x^2} + \frac{1}{3}\,\frac{d(a^2 - x^2)}{a^2 - x^2},$$

from which the value of dy may be easily found. This method of passing to logarithms may often be resorted to with advantage.

13. Differentiate $y = \dfrac{1 + x}{1 + x^2}$.

We have

$$dy = \frac{(1 + x^2)\,dx - (1 + x)2x\,dx}{(1 + x^2)^2} = \frac{1 - 2x - x^2}{(1 + x^2)^2}dx.$$

14. Differentiate $\dfrac{1}{(a + x)^2(b + x)^3}$.

We shall find

$$dy = -\frac{2(b + x) + 3(a + x)}{(a + x)^3(b + x)^4}dx.$$

15. Differentiate $y = (a+x)^m(b+x)^n$.

We have

$$\log y = m\log(a+x) + n\log(b+x).$$

$$\therefore \quad \frac{dy}{y} = \frac{m\,d(a+x)}{a+x} + \frac{n\,d(b+x)}{b+x};$$

$$\therefore \quad dy = (a+x)^m(b+x)^n\left\{\frac{m}{a+x} + \frac{n}{b+x}\right\}dx.$$

16. Differentiate $y = \dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$.

$$\text{Ans. } dy = \frac{\sqrt{a}(\sqrt{x}-\sqrt{a})}{2\sqrt{x}\sqrt{a+x}(\sqrt{a}+\sqrt{x})^2}\,dx.$$

17. Differentiate $y = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$.

Passing to logarithms, we have

$$\log y = \tfrac{1}{2}\log(1+x) - \tfrac{1}{2}\log(1-x).$$

$$\therefore \quad dy = \frac{dx}{(1-x)\sqrt{1-x^2}}$$

18. Differentiate $y = \{x + \sqrt{1-x^2}\}^n$.

We have

$$dy = n\{x+\sqrt{1-x^2}\}^{n-1}d\{x+\sqrt{1-x^2}\} = ?$$

19. Differentiate $y = \log\dfrac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}$.

We have

$$y = \log\{\sqrt{1+x^2}-x\} - \log\{\sqrt{1+x^2}+x\}.$$

$$\therefore \quad dy = ?$$

20. Differentiate $y = a^{b+x}$.

We have

$$dy = a^{b+x}\log a\,dx.$$

21. Differentiate $y = x^x$.

We have

$$\log y = x \log x.$$

$$\therefore \ \frac{dy}{y} = \log x.\ dx + dx. \quad \therefore \ dy = x^x(\log x + 1)\ dx.$$

22. Differentiate $y = x^{x^x}$, the notation signifying the x^x power of x.

We have

$$\log y = x^x \log x.$$

$$\therefore \ dy = x^{x^x}\, x^x \left\{ \log x(\log x + 1) + \frac{1}{x} \right\} dx.$$

23. Differentiate $y = e^{x^n}$.

Ans. $dy = n\, e^{x^n}\, x^{n-1}\, dx.$

24. Differentiate $y = e^{x^x}$.

Ans. $dy = e^{x^x}\, x^x(\log x + 1)\ dx.$

25. Differentiate $y = x^{e^x}$.

26. Differentiate $y = \log(\log x)$.

We have

$$dy = \frac{d \log x}{\log x} = \frac{dx}{x \log x}.$$

27. Differentiate $y = (\log x)^n$.

Ans. $dy = n(\log x)^{n-1} \dfrac{dx}{x}.$

28. Differentiate $y = e^{\log x}$.

29. Differentiate $y = e^{\log \sqrt{a^2+x^2}}$.

30. Differentiate $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Ans. $dy = \dfrac{4}{(e^x + e^{-x})^2}\ dx.$

31. Differentiate $y = \frac{x}{e^x - 1}$.

32. Differentiate $y = e^{\log^n x}$.

33. Differentiate $y = x^5 \log^3 x + 2x^3 \log^4 x + 4x\, e^{2x}$.

34. Differentiate $y = e^x \sqrt{\frac{1+x}{1-x}}$.

$$\text{Ans.}\quad dy = e^x \frac{2 - x^2}{(1-x)\sqrt{1-x^2}}\, dx.$$

35. Differentiate $y = 4 \sin^5 x$.

We have

$$dy = 20 \sin^4 x\, d \sin x = 20 \sin^4 x.\ \cos x\, dx.$$
$$= 10 \sin^3 x.\ 2 \sin x.\ \cos x\, dx = 10 \sin^3 x.\ \sin 2x\, dx.$$

36. Differentiate $y = \sin nx$.

37. Differentiate $y = \operatorname{tang}^n x$.

38. Differentiate $y = \sin 3x.\ \cos 2x$.

We have

$$dy = \sin 3x\, d \cos 2x + \cos 2x\, d \sin 3x$$
$$= -2 \sin 3x.\ \sin 2x\, dx + 3 \cos 3x.\ \cos 2x\, dx$$
$$= (\cos 3x.\ \cos 2x + 2 \cos 5x)\ \ dx.$$

39. Differentiate $y = \log \sin x$; $y = \log \cos x$; $y = \log \operatorname{tang} x$; $y = \log \cot x$; $y = \log \sec x$; $y = \log \operatorname{cosec} x$; $y = \log \operatorname{versin} x$; $y = \log \operatorname{coversin} x$.

40. Differentiate $y = \sin^x x$.

41. Differentiate $y = \sin(\sin x)$.

We have

$$dy = \cos(\sin x)\, d \sin(x)$$
$$= \cos(\sin x).\ \cos x\, dx.$$

42. Differentiate $y = (\cos x)^{\sin x}$.

We have

$$\log y = \sin x.\ \log \cos x.$$

$$\therefore \frac{dy}{y} = \cos x\, dx.\ \log \cos x + \sin x \frac{d \cos x}{\cos x}$$

$$= \frac{\cos^2 x.\ \log \cos x - \sin^2 x}{\cos x} dx;$$

and $dy = (\cos x)^{\sin x - 1} \{\cos^2 x.\ \log \cos x - \sin^2 x\}\, dx.$

43. Differentiate $y = \log \left\{ \dfrac{a + b \operatorname{tang} x}{a - b \operatorname{tang} x} \right\}.$

44. Differentiate $y = \sin^{-1} \dfrac{x}{\sqrt{1 + x^2}}.$

We have

$$dy = \frac{d\left\{\frac{x}{\sqrt{1+x^2}}\right\}}{\sqrt{1 - \frac{x^2}{1+x^2}}} = ?$$

45. Differentiate $y = \sin^{-1}(\sin x).$

We have

$$dy = \frac{d \sin x}{\sqrt{1 - \sin^2 x}} = \frac{\cos x\, dx}{\cos x} = dx.$$

46. Differentiate $y = \log \sqrt{\sin x} + \log \sqrt{\cos x}.$

47. Differentiate $y = \log\{\cos x + \sqrt{-1} \sin x\}.$

Ans. $\dfrac{dy}{dx} = \sqrt{-1}.$

48. Differentiate $y = \dfrac{e^{ax}(a \sin x - \cos x)}{a^2 + 1}.$

We have

$$dy = \frac{dx}{a^2 + 1} \{a\, e^{ax}(a \sin x - \cos x) + ae^{ax} \cos x + e^{ax} \sin x\}$$

$$= e^{ax} \sin x\, dx.$$

49. Differentiate $y = \cos^{-1} \left\{ \dfrac{a \cos x + b}{b \cos x + a} \right\}.$

38. In the next article we present some examples to be solved by the methods for functions of functions and compound functions. For convenience we repeat the formulas for these cases.

1. If $u = F(y)$ and $y = f(x)$; $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$.

2. If $u = F(y.\, z\, .\, .\, .)$, $y = f(x)$, $z = \phi(x)$ . . . ;

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} \dots$$

39. Ex. 1.—Given $u = a^y$ and $y = b^x$; find $\frac{du}{dx}$.

We have

$$\frac{du}{dy} = a^y \log a\,; \quad \frac{dy}{dx} = b^x \log b.$$

$$\therefore \quad \frac{du}{dx} = a^y\, b^x \log a .\, \log b.$$

2. Given $u = \log y$, $y = \log x$; find $\frac{du}{dx}$.

We have

$$\frac{du}{dy} = \frac{1}{y}; \quad \frac{dy}{dx} = \frac{1}{x}.$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{x \log x}.$$

3. Given $u = \sqrt{z}$, $z = x + (x + a)^{\frac{1}{2}}$; find $\frac{du}{dx}$.

4. Given $u = \text{tang}^{-1} y$, $y = \frac{2ax - x^2}{a^2 + x^2}$; find $\frac{du}{dx}$.

5. Given $u = \sin^{-1} y$, $y = e^x \cos x$; find $\frac{du}{dx}$.

6. Given $u = e^y$, $y = \text{tang}^{-1} x$; find $\frac{du}{dx}$.

7. Given $u = x^y$, $y = x^{x^x}$; find $\frac{du}{dx}$.

8. Given $u = \sec^{-1} y$, $y = \frac{1}{\cos x}$; find $\frac{du}{dx}$.

9. Given $u = \sin^{-1}(y - z)$, $y = 3x$, $z = 4x^3$; find $\frac{du}{dx}$.

We have

$$\frac{du}{dy} = \frac{1}{\sqrt{1-(y-z)^2}}; \quad \frac{du}{dz} = \frac{-1}{\sqrt{1-(y-z)^2}};$$

$$\frac{dy}{dx} = 3; \quad \frac{dz}{dx} = 12x^2.$$

$$\therefore \frac{du}{dx} = \frac{3 - 12x^2}{\sqrt{1-(3x-4x^3)^2}}.$$

10. Given $u = yv$, $y = e^x$, $v = x^x$; find $\frac{du}{dx}$.

11. Given $u = \frac{y}{z}$, $y = e^x - e^{-x}$, $z = e^x + e^{-x}$; find $\frac{du}{dx}$.

12. Given $u = \operatorname{tang}^{-1} y$, $y = x + \sqrt{1-x^2}$; find $\frac{du}{dx}$.

Ans. $\frac{du}{dx} = \frac{\sqrt{1-x^2} - x}{2\sqrt{1-x^2}(1 + x\sqrt{1-x^2})}$.

13. Given $u = \operatorname{tang}^{-1} y$, $y = \sqrt{\frac{1-\cos x}{1+\cos x}}$; find $\frac{du}{dx}$.

Ans. $\frac{du}{dx} = \frac{1}{2}$.

14. Given $u = \log y$, $y = \sqrt{1+x} + \sqrt{1-x}$; find $\frac{du}{dx}$.

15. Given $y^2 - 2axy + x^2 - b^2 = 0$; find $\frac{dy}{dx}$.

Assume $u = y^2 - 2axy + x^2 - b^2 = 0$.

Then we have

$$u = F(x.\ y) = 0, \text{ and (Art. 34 [4])}$$

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0. \tag{1}$$

$$\text{Now,}\quad \frac{du}{dx} = 2x - 2ay;\quad \frac{du}{dy} = 2y - 2ax.$$

$\therefore$ by substitution in (1) and reduction,

$$\frac{dy}{dx} = \frac{ay - x}{y - ax}.$$

It will be noticed that equation (1) in this example is the same as the formula for implicit functions in Art. 34.

16. Given $u = y^3 + 3axy + x^3 = 0$; to find $\frac{dy}{dx}$.

Employing equation (1) as above, we have

$$\frac{du}{dx} = 3x^2 + 3ay;\quad \frac{du}{dy} = 3y^2 + 3ax;$$

$$\therefore\quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}.$$

17. Given $u = y^4 + 4y^3x + 6y^2x^2 + 4yx^3 + x^4 = 0$; to find $\frac{dy}{dx}$.

CHAPTER IV.

SUCCESSIVE DIFFERENTIATION.

40. The derivative of a function of x will, in general, be itself a function of x, and will therefore have its derivative and differential, which may, in their turn, be differentiated; and so on.

The process of obtaining the derivatives and differentials of derivatives and differentials is called **successive differentiation.**

In determining the successive derivatives of a function, we shall suppose the differential of the independent variable

to maintain a *constant* value, as we are evidently at liberty to do, and this will introduce great simplicity into the operation.

41. Notation.—Let $y = F(x)$, and $\frac{dy}{dx} = F'(x)$.

By differentiation we obtain

$$\frac{d\left\{\frac{dy}{dx}\right\}}{dx} = F''(x),$$

designating by $F''(x)$ the derivative of $F'(x)$.

Differentiating again, we have

$$\frac{d\left\{\frac{d\left\{\frac{dy}{dx}\right\}}{dx}\right\}}{dx} = F'''(x);$$

and so on.

The notation used in the first members of the above equations being inconvenient, it is in practice replaced by the following:

$$d(dy) \text{ is denoted by } d^2y; \quad d\{d(dy)\} \text{ by } d^3y;$$
$$dx.\,dx \text{ by } dx^2; \quad dx.\,dx.\,dx \text{ by } dx^3;$$

and so on.

The above equations will then become

$$y = F(x); \quad \frac{dy}{dx} = F'(x); \quad \frac{d^2y}{dx^2} = F''(x); \quad \frac{d^3y}{dx^3} = F'''(x); \quad \frac{d^ny}{dx^n} = F^n(x),$$

in which

$F'(x)$, $F''(x)$ . . . $F^n(x)$ are the 1st, 2d . . . nth *derivatives* of $F(x)$ or y;

dy, d^2y . . . d^ny are the 1st, 2d . . . nth *differentials* of $F(x)$ or y;

dx, dx^2 . . . dx^n are the 1st, 2d . . . nth *powers* of the differential of x;

$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ . . . $\frac{d^ny}{dx^n}$ are the 1st, 2d . . . nth *differential co-efficients of* $F(x)$ or y, with respect to x.

The student must be careful not to confound $d^n y$ with dy^n.

42. EXAMPLES.

1. $y = x^n$; $\frac{dy}{dx} = n\,x^{n-1}$; $\frac{d^2y}{dx^2} = n(n-1)\,x^{n-2}$;

$\frac{d^3y}{dx^3} = n(n-1)(n-2)\,x^{n-3}$; $\frac{d^ny}{dx^n} = n(n-1)(n-2)\ldots 3.\,2.\,1.$

2. $y = \log x$; $\frac{dy}{dx} = \frac{1}{x}$; $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$; $\frac{d^3y}{dx^3} = \frac{2}{x^3}$; $\frac{d^ny}{dx^n} = ?$

3. $y = \sin x$; $\frac{dy}{dx} = \cos x$; $\frac{d^2y}{dx^2} = -\sin x$; $\frac{d^3y}{dx^3} = -\cos x$;

$\frac{d^4y}{dx^4} = \sin x$; etc.

4. $y = a^x$; $\frac{dy}{dx} = a^x \log a$; $\frac{d^2y}{dx^2} = a^x \log^2 a$;

$\frac{d^3y}{dx^3} = a^x \log^3 a$; etc.

5. $y = e^x$; $\frac{dy}{dx} = e^x$; $\frac{d^2y}{dx^2} = e^x$; $\frac{d^3y}{dx^3} = e^x$; etc.

6. $y = u\,v$, in which u and v are functions of x.
We have already found

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Differentiating both sides of this equation with respect to x, and observing that $\frac{dv}{dx}$, $\frac{du}{dx}$ are functions of x, we have

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\cdot\frac{du}{dx}$$

$$= u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}.$$

Differentiating again, we have

$$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + \frac{du}{dx}\cdot\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{d^2v}{dx^2} + 2\frac{d^2u}{dx^2}\cdot\frac{dv}{dx}$$

$$+ \frac{dv}{dx}\cdot\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}$$

$$= u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\cdot\frac{d^2v}{dx^2} + 3\frac{dv}{dx}\cdot\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}.$$

Similarly we shall find

$$\frac{d^4y}{dx^4} = u\frac{d^4v}{dx^4} + 4\frac{du}{dx}\cdot\frac{d^3v}{dx^3} + 6\frac{d^2u}{dx^2}\cdot\frac{d^2v}{dx^2} + 4\frac{dv}{dx}\cdot\frac{d^3u}{dx^3} + v\frac{d^4u}{dx^4}.$$

A simple inspection of the foregoing results shows that the co-efficients follow the same law as in the Binomial formula, and it may readily be shown that this will always be the case. The resulting formula for $\frac{d^ny}{dx^n}$ is known as **Leibnitz' Theorem.**

43. We give here two examples in the successive differentiation of implicit functions of a single variable, reserving until a subsequent chapter the notation and formulas adopted for such cases.

1. $u = x^3 + 3axy + y^3 = 0.$

We have already found [Ex. 16, Art. 39,]

$$\frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}.$$

Designating $\frac{dy}{dx}$ by u', we shall have, since u' is a function of x directly and also indirectly through y,

$$\frac{d^2y}{dx^2} = \frac{du'}{dx} = \frac{du'}{dy}\cdot\frac{dy}{dx} + \frac{du'}{dx}. \qquad (1)$$

Now, $$\frac{du'}{dx} = \frac{-2x(y^2 + ax) + a(x^2 + ay)}{(y^2 + ax)^2};$$

$$\text{and } \frac{du'}{dy} = \frac{-a(y^2+ax)+2y(x^2+ay)}{(y^2+ax)^2}.$$

$\therefore$ by substitution in (1) and reduction,

$$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(y^2+ax)^3}.$$

2. $u = y^3 - 3y + x = 0.$

We shall find

$$\frac{dy}{dx} = \frac{1}{3(1-y^2)}.$$

Placing this equal to u', and observing that u' is a function of x through y, we have

$$\frac{du'}{dx} = \frac{d^2y}{dx^2} = \frac{du'}{dy} \cdot \frac{dy}{dx} \qquad (1).$$

$$\text{Now, } \frac{du'}{dy} = \frac{2y}{3(1-y^2)^2}.$$

$\therefore$ by substitution in (1),

$$\frac{d^2y}{dx^2} = \frac{2y}{9(1-y^2)^3}.$$

We may also obtain the value of $\frac{d^2y}{dx^2}$ as follows:

Expanding the value of $\frac{dy}{dx}$ by actual division, we have

$$\frac{dy}{dx} = \tfrac{1}{3}(1+y^2+y^4+\text{etc.})$$

$$\therefore \frac{d^2y}{dx^2} = \tfrac{1}{3}(2y+4y^3+\text{etc.})\frac{dy}{dx} = \frac{1}{9}(2y+6y^3+\text{etc.})$$

44. Problem.—*To find an expression for the ratio of any finite increments of two functions of the same variable.*

Let $F(x)$, $f(x)$ be two functions of x, and let x_0, X be two arbitrary values of x, such that

$$X = x_0 + h.$$

It is required to find an expression for the ratio

$$\frac{F(X) - F(x_0)}{f(X) - f(x_0)} \text{ or } \frac{F(x_0 + h) - F(x_0)}{f(x_0 + h) - f(x_0)}.$$

Let us suppose that the derivative $f'(x)$ is positive for all values of x from x_0 to X. Also, let A and B be greater than the greatest and less than the least values of the ratio $\frac{F'(x)}{f'(x)}$ between x_0 and X.

Then we shall have

$$\frac{F'(x)}{f'(x)} < A; \quad \frac{F'(x)}{f'(x)} > B;$$

and therefore

$$F'(x) < Af'(x); \quad F'(x) > Bf'(x).$$

Now, $F'(x)$, $Af'(x)$, $Bf'(x)$, are the derivatives of $F(x)$, $Af(x)$, $Bf(x)$. Consequently [Art. 18, Cor.], $F(x)$ increases less rapidly than $Af(x)$, but more rapidly than $Bf(x)$.

Therefore,

$$F(X) - F(x_0) < A\{f(X) - f(x_0)\}, \text{ and} > B\{f(X) - f(x_0)\}.$$

$$\therefore \frac{F(X) - F(x_0)}{f(X) - f(x_0)} < A, \text{ and} > B.$$

If, then, the ratio $\frac{F'(x)}{f'(x)}$ be continuous between the values x_0 and X, which will be the case if $F'(x)$, $f'(x)$ are continuous, there must be between x_0 and X some value of x which will render this ratio exactly equal to $\frac{F(X) - F(x_0)}{f(X) - f(x_0)}$. Call-

ing this value of x, $x_0 + \theta h$ (θ being less than unity), we have

$$\frac{F(X) - F(x_0)}{f(X) - f(x_0)} = \frac{F'(x_0 + \theta h)}{f'(x_0 + \theta h)}. \qquad (1).$$

If we had taken $f'(x)$ negative, the inequalities in the above demonstration would have been reversed, and we would have been led in the same manner to the equation (1). This equation is therefore general, provided $f'(x)$ retains the same sign for all values of x between x_0 and $x_0 + h$.

45. Let us suppose that for x_0 we have $F(x_0) = 0$, $f(x_0) = 0$; then, designating θh by h', we shall have

$$\frac{F(x_0 + h)}{f(x_0 + h)} = \frac{F'(x_0 + h')}{f'(x_0 + h')}.$$

If at the same time $F'(x_0) = 0$, $f'(x_0) = 0$, then we shall have in the same way

$$\frac{F'(x_0 + h')}{f'(x_0 + h')} = \frac{F''(x_0 + h'')}{f''(x_0 + h'')},$$

h'' being less than h', and $\dfrac{F''(x)}{f''(x)}$ being continuous; hence,

$$\frac{F(x_0 + h)}{f(x_0 + h)} = \frac{F''(x_0 + h'')}{f''(x_0 + h'')}.$$

Continuing thus, we shall find that if

$$F(x_0) = 0, F'(x_0) = 0 \,..\, F^{n-1}(x_0) = 0, f(x_0) = 0, f'(x_0) = 0 \,..\, f^{n-1}(x_0) = 0,$$

and the ratios $\dfrac{F(x)}{f(x)}, \dfrac{F'(x)}{f'(x)}, \ldots \dfrac{F^{n-1}(x)}{f^{n-1}(x)}$, are continuous, then

$$\frac{F(x_0 + h)}{f(x_0 + h)} = \frac{F^n(x_0 + \theta h)}{f^n(x_0 + \theta h)} \qquad (2),$$

in which θ represents some positive quantity less than unity.

Corollary 1.—If all the foregoing conditions be fulfilled except $F(x_0)=0$, then

$$\frac{F(x_0+h)-F(x_0)}{f(x_0+h)}=\frac{F^n(x_0+\theta h)}{f^n(x_0+\theta h)} \qquad (3).$$

Corollary 2.—If $x_0=0$, then

$$\frac{F(h)}{f(h)}=\frac{F^n(\theta h)}{f^n(\theta h)},$$

or, writing x for h,

$$\frac{F(x)}{f(x)}=\frac{F^n(\theta x)}{f^n(\theta x)} \qquad (4),$$

an equation which leads to the following

Theorem.—*If two continuous functions of* x, *together with their first* n — 1 *derivatives, are zero for* x = 0, *and if the first* n *derivatives of one of them have the same sign for all values of* x *between* x_0 *and* X; *then the ratio of the functions will be equal to that of their* n^{th} *derivatives, in both of which the value of* x *is some value between* x_0 *and* X.

46. The conditions which have been imposed upon $f(x)$ in the foregoing equations will evidently be fulfilled if we take

$$f(x)=(x-x_0)^n;$$

for, we shall have

$$f(x_0)=0;\ f'(x_0)=0;\ \ldots f^{n-1}(x_0)=0;\ f^n(x_0)=1.\ 2.\ 3\ \ldots$$
$$\ldots (n-1)\, n;\ f(x_0+h)=f(h)=h^n.$$

Hence, equation (3) becomes

$$F(x_0+h)-F(x_0)=\frac{h^n}{1.\ 2\ \ldots\ n}\,F^n(x_0+\theta h) \qquad (5);$$

and if $F(x_0)=0$, then

$$F(x_0+h)=\frac{h^n}{1.\,2\,.\,.\,.\,n}\,F^n(x_0+\theta h) \qquad (6).$$

If, also, $x_0=0$, then

$$F(h)=\frac{h^n}{1.\,2\,.\,.\,.\,n}\,F^n(\theta h),$$

$$\text{or}\quad F(x)=\frac{x^n}{1.\,2\,.\,.\,.\,n}\,F^n(\theta x) \qquad (7).$$

If $F(x_0)$ be not zero for $x_0=0$, then

$$F(x)-F(x_0)=\frac{x^n}{1.\,2\,.\,.\,.\,n}\,F^n(\theta x) \qquad (8).$$

47. Corollary 1.—If $\frac{F(x)}{x^{n-1}}$ tends toward zero at the same time with x, then

$$\frac{F(x)}{x^n}=\frac{F^n(\theta x)}{1.\,2\,.\,.\,.\,n}.$$

For, we must in such case have $F(0)=0$, $F'(0)=0\,.\,.\,.\,F^{n-1}(0)=0$; otherwise, $\frac{F(x)}{x^{n-1}}$ would be infinite for $x=0$. The conditions attached to equation (7) are consequently satisfied, and hence $\frac{F(x)}{x^n}=\frac{F^n(\theta x)}{1.\,2\,.\,.\,.\,n}$.

Corollary 2.—If in (5) we make $n=1$, and replace x_0 by x, we shall have

$$F(x+h)-F(x)=h\,F'(x+\theta h) \qquad (9).$$

From this we infer that *if the derivative of an expression taken with respect to* x *is zero for every value of* x, *the value of the expression is independent of* x.

For, let $F(x)$ be such an expression; then we shall have

$$h\,F'(x+\theta h)=0, \text{ and } F(x+h)-F(x)=0,$$

$$\text{or } F(x+h)=F(x).$$

Therefore $F(x)$ has the same value for every value of x, which can only be the case when it is independent of x.

Also, *if two functions of* x *have the same derivative with respect to* x, *they differ from each other by a constant.*

For, the derivative of their difference being equal to the difference of their derivatives, and this being *zero*, their difference must be independent of x, and therefore constant.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO ANALYSIS.

CHAPTER V.

TAYLOR'S AND MACLAURIN'S FORMULAS.

48. The preceding chapters contain the most important principles of the Differential Calculus, so far as it relates to functions of a single variable. We propose now to exhibit some applications of the theory, beginning with certain useful formulas for the development of functions into series.

49. Taylor's Formula

has for its object the development of F(x + h) *in terms of the ascending powers of* h.

This formula may be demonstrated as follows. From equation (9), Art. 47, we have

$$F(x+h)-F(x)=h\,F'(x+\theta h),$$

the only condition of which equation is that $F(x)$ and its

derivative $F'(x)$ are continuous between the limits x and $x+h$.

This equation may be written

$$F(x+h)-F(x)=h\ F'(x)+R_1,$$

$$\text{or}\quad F(x+h)-F(x)-h\ F'(x)=R_1;$$

R_1 being a function of h which vanishes when $h=0$.

Now, the first derivative of R_1 is evidently *zero* when $h=0$, and its second derivative is $F''(x+h)$.

We therefore have [Art. 46]

$$R_1=\frac{h^2}{1.\ 2}F''(x+\theta h),$$

and consequently,

$$F(x+h)-F(x)-h\ F'(x)=\frac{h^2}{1.\ 2}\ F''(x+\theta h),$$

which may be written

$$F(x+h)-F(x)-h\ F'(x)=\frac{h^2}{1.\ 2}\ F''(x)+R_2,$$

$$\text{or}\quad F(x+h)-F(x)-h\ F'(x)-\frac{h^2}{1.\ 2}\ F''(x)=R_2.$$

R_2 is a function of h which, together with its first and second derivatives, vanishes when $h=0$, as may be readily seen, and its third derivative is $F'''(x+h)$.

Hence [Art. 46],

$$R_2=\frac{h^3}{1.\ 2.\ 3}\ F'''(x+\theta h);$$

$$\text{and}\ F(x+h)-F(x)-h\ F'(x)-\frac{h^2}{1.\ 2}\ F''(x)=\frac{h^3}{1.\ 2.\ 3}\ F'''(x+\theta h),$$

or $F(x+h) = F(x) + h\,F'(x) + \frac{h^2}{1.\,2}\,F''(x) + \frac{h^3}{1.\,2.\,3}\,F'''(x+\theta h)$.

Continuing indefinitely in this manner, we shall find

$$F(x+h) = F(x) + h\,F'(x) + \frac{h^2}{1.\,2}\,F''(x) + \frac{h^3}{1.\,2.\,3}\,F'''(x) + \dots$$
$$+ \frac{h^{n-1}}{1.\,2\dots(n-1)}\,F^{n-1}(x) + \frac{h^n}{1.\,2\dots n}\,F^n(x+\theta h) \qquad (1).$$

If now the last term of this equation tends toward *zero* as n increases, then

$F(x+h)$ is the limit to the sum of the terms of the series $F(x),\ hF'(x),\ \frac{h^2}{1.\,2}F''(x)\ \dots$, and we may write

$$F(x+h) = F(x) + h\,F'(x) + \frac{h^2}{1.\,2}\,F''(x) + \frac{h^3}{1.\,2.\,3}\,F'''(x) + \dots \text{ etc.} \qquad (2).$$

If we designate $F(x)$ by y, $F(x+h)$ by y', and substitute for the derivatives $F'(x)$, $F''(x)$, etc., the differential coefficients which are respectively equal to them, we shall have

$$F(x+h) = y' = y + h\frac{dy}{dx} + \frac{h^2}{1.\,2}\cdot\frac{d^2y}{dx^2} + \frac{h^3}{1.\,2.\,3}\cdot\frac{d^3y}{dx^3} + \text{etc.} \qquad (3).$$

Equations (2) and (3) are, both, forms of Taylor's formula.

NOTE.—It must be borne in mind that this formula depends upon the conditions that $F(x)$ and all of its derivatives are continuous between the limits x and $x+h$, and that $\frac{h^n}{1.\,2\dots n}\,F^n(x+\theta h)$ tends toward *zero* as n increases. If the last condition is not satisfied, then formula (1) gives the development of $F(x+h)$.

50. The term $\frac{h^n}{1.\,2\dots n}\,F^n(x+\theta h)$ tends toward *zero* as n increases whenever $F^n(x)$ is finite for all values of n. For,

under this supposition, whatever may be the value of h, as soon as n passes this value, the co-efficient $\frac{h^n}{1.2\ldots n}$ will be multiplied (in finding the next succeeding terms) by the factors $\frac{h}{h+1}$, $\frac{h}{h+2}$, etc., which form a continually decreasing series; and the limit to the product of these factors, and therefore to the term containing that product, will obviously be *zero.*

Therefore, *whenever* F(x) *and all of its derivatives are finite and continuous, Taylor's formula gives the exact development of* F(x + h).

51. Formula (1) of Art. 49 imposes no condition upon the derivatives which are of an order superior to the n^{th}. These may be *discontinuous* for certain values of x between the extremes x and $x + h$, and the *formula* be still exact; and thus the *development* by this formula may, in a given case, be exact up to a certain term, and inexact beyond that term.

For example let us take

$$F(x) = f(x) + (x - x_0)^{m+\frac{p}{q}}\,\phi(x).$$

The derivatives of this expression for the particular value x_0 of x will be finite as far as $F^m(x_0)$ if those of $f(x)$ and $\phi(x)$ are so, but beyond this term they will be *infinite.* The development will then be exact only up to the term preceding that which contains $F^m(x_0)$, and in order to render it complete it will be necessary to add a term corresponding to the last term of formula (1), which depends upon the m^{th} derivative.

Taylor's formula will, therefore, in general terms, *give the exact development of* F(x + h) *only up to the first derivative which becomes* infinite *for any value of* x *between the limits* x *and* x + h.

By means of formula (1) we may determine the limits of the error committed in stopping the development at any

given term of Taylor's formula. In fact, if we take the first n terms, the exact quantity which it is necessary to add in order to obtain the value of $F(x+h)$ is

$$\frac{h^n}{1.\,2\,.\,.\,.\,n} F^n(x+\theta h).$$

If, then, we designate by A and B the smallest and greatest values of $F^n(x)$ between x and $x+h$, the error committed in taking the first n terms will be between $\frac{Ah^n}{1.\,2\,.\,.\,.\,n}$ and $\frac{Bh^n}{1.\,2\,.\,.\,.\,n}$.

52. Maclaurin's Formula.

Maclaurin's formula has for its object the development of a function of a single variable in terms of the ascending powers of that variable.

If in formula (1), Art. 49, we make $x=0$, and then substitute x for h, we shall have

$$F(x)=F(0)+x\,F'(0)+\frac{x^2}{1.\,2}F''(0)+\,.\,.\,.\,+\frac{x^n}{1.\,2\,.\,.\,.\,n}F^n(\theta x) \quad (4).$$

If, now, as n increases, the last term tends toward *zero*, $F(x)$ is the limit to the sum of the terms $F(0)$, $x\,F'(0)$, etc., and we may write

$$F(x)=F(0)+x\,F'(0)+\frac{x^2}{1.\,2}F''(0)+\,.\,.\,.\,\text{etc.} \quad (5).$$

If $F(x)$ be designated by y, and we indicate, by inclosing them in brackets, the values assumed by $F(x)$, $F'(x)$, etc., when $x=0$, then (5) may be written

$$F(x)=y=[y]+x[F'(x)]+\frac{x^2}{1.\,2}[F''(x)]+\,.\,.\,.\,\text{etc.} \quad (6);$$

or, replacing the derivatives by the differential co-efficients,

$$F(x)=y=[y]+x\left[\frac{dy}{dx}\right]+\frac{x^2}{1.\ 2}\left[\frac{d^2y}{dx^2}\right]+\text{etc.} \qquad (7).$$

Equations (5), (6), and (7) are different, but equivalent, forms of Maclaurin's formula.

This formula is attended with limitations similar to that of Taylor. It ceases to give the exact development of a function, whenever that function or any of its derivatives become infinite for finite values of x; but it will be observed that while Taylor's formula may fail for but a single value of x between x and $x+h$, if Maclaurin's formula fails for one value of x it fails for all other values.

53. EXAMPLES.

1. Develop $y=\sin x$.

Forming the successive derivatives, we have

$$F'(x)=\cos x;\ F''(x)=-\sin x;\ F'''(x)=-\cos x;\ F^{\text{iv}}(x)=\sin x, \text{ etc.}$$

Making $x=0$, these become $F(0)=\sin 0=0$;

$$F'(0)=\cos 0=1;\ F''(0)=-\sin 0=0;\ F'''(0)=-\cos 0 \\ =-1;\ F^{\text{iv}}(0)=\sin 0=0;\ \text{etc.}$$

Therefore, by substitution in Maclaurin's formula, which is applicable here since none of the derivatives are infinite, we have

$$y=\sin x=x-\frac{x^3}{1.\ 2.\ 3}+\frac{x^5}{1.\ 2.\ 3.\ 4.\ 5}-\frac{x^7}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}+\text{etc.}$$

2. Develop $y=\cos x$.

The derivatives are

$$F'(x)=-\sin x;\ F''(x)=-\cos x;\ F'''(x)=\sin x; \\ F^{\text{iv}}(x)=\cos x;\ \text{etc.}$$

Making $x=0$, we have

$$F(0)=\cos 0=1;\ F'(0)=-\sin 0=0;$$

$$F''(0)=-\cos 0=-1;\ F'''(0)=\sin 0=0;$$
$$F^{\text{iv}}(0)=\cos 0=1;\ \text{etc.}$$

Therefore, by substitution in Maclaurin's formula,

$$y=\cos x=1-\frac{x^2}{1.\,2}+\frac{x^4}{1.\,2.\,3.\,4}-\frac{x^6}{1.\,2.\,3.\,4.\,5.\,6}+\text{etc.}$$

3. Develop $y=a^x$.
We have

$$F(x)=a^x;\ F'(x)=a^x\log a;\ F''(x)=a^x\log^2 a;$$
$$F'''(x)=a^x\log^3 a;\ \text{etc.}$$

$\therefore$, making $x=0$,

$$F(0)=a^0=1;\ F'(0)=\log a;\ F''(0)=\log^2 a;$$
$$F'''(0)=\log^3 a;\ \text{etc.}$$

Therefore, by Maclaurin's formula,

$$y=a^x=1+\frac{x}{1}\log a+\frac{x^2}{1.\,2}\log^2 a+\frac{x^3}{1.\,2.\,3}\log^3 a+\text{etc.}$$

Corollary 1.—If $a=e$, then $\log a=\log e=1$, and

$$\therefore\ e^x=1+\frac{x}{1}+\frac{x^2}{1.\,2}+\frac{x^3}{1.\,2.\,3}+\text{etc.}$$

Corollary 2.—If $x=1$, then $e^x=e$, and

$$\therefore\ e=1+1+\frac{1}{1.\,2}+\frac{1}{1.\,2.\,3}+\text{etc.},$$

a formula for the base of the Naperian system of logarithms.

4. **Euler's Formulas.**—If in the series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.\ 2} + \frac{x^3}{1.\ 2.\ 3} + \text{etc.},$$

we put $x = z\sqrt{-1}$, and $x = -z\sqrt{-1}$, we shall have

$$e^{z\sqrt{-1}} = 1 + \frac{z\sqrt{-1}}{1} - \frac{z^2}{1.\ 2} - \frac{z^3\sqrt{-1}}{1.\ 2.\ 3} + \frac{z^4}{1.\ 2.\ 3.\ 4}, \text{ etc.},$$

$$e^{-z\sqrt{-1}} = 1 - \frac{z\sqrt{-1}}{1} - \frac{z^2}{1.\ 2} + \frac{z^3\sqrt{-1}}{1.\ 2.\ 3} + \frac{z^4}{1.\ 2.\ 3.\ 4}, \text{ etc.}$$

Adding and subtracting these two equations, we have

$$e^{z\sqrt{-1}} + e^{-z\sqrt{-1}} = 2\left(1 - \frac{z^2}{1.\ 2} + \frac{z^4}{1.\ 2.\ 3.\ 4}, \text{ etc.}\right) = 2 \cos z;$$

$$e^{z\sqrt{-1}} - e^{-z\sqrt{-1}} = 2\sqrt{-1}\left(z - \frac{z^3}{1.\ 2.\ 3} + \frac{z^5}{1.\ 2.\ 3.\ 4.\ 5}, \text{ etc.}\right)$$
$$= 2\sqrt{-1} \sin z.$$

$$\therefore \sin z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}; \qquad \cos z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2};$$

and by division,

$$\frac{\sin z}{\cos z} = \text{tang } z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{\sqrt{-1}\{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}\}}.$$

These are Euler's formulas for the sine, cosine, and tangent of an arc in terms of imaginary exponentials.

5. **Demoivre's Formula.**—Resuming the equations of the last example,

$$e^{z\sqrt{-1}} = 1 + \frac{z\sqrt{-1}}{1} - \frac{z^2}{1.\ 2} - \frac{z^3\sqrt{-1}}{1.\ 2.\ 3} + \frac{z^4}{1.\ 2.\ 3.\ 4}, \text{ etc.},$$

$$e^{-z\sqrt{-1}} = 1 - \frac{z\sqrt{-1}}{1} - \frac{z^2}{1.\ 2} + \frac{z^3\sqrt{-1}}{1.\ 2.\ 3} + \frac{z^4}{1.\ 2.\ 3.\ 4}, \text{ etc.},$$

we notice that, by Exs. 1 and 2, the second members of these equations are respectively equal to

$$\cos z + \sqrt{-1} \sin z \text{ and } \cos z - \sqrt{-1} \sin z.$$

$\therefore$ substituting mx for z, we have

$$e^{\pm mx\sqrt{-1}} = \cos mx \pm \sqrt{-1} \sin mx.$$

But $e^{\pm mx\sqrt{-1}} = \{e^{\pm x\sqrt{-1}}\}^m = \{\cos x \pm \sqrt{-1} \sin x\}^m$, as above.

Hence,

$$\cos mx \pm \sqrt{-1} \sin mx = \{\cos x \pm \sqrt{-1} \sin x\}^m.$$

This formula serves to convert powers of sines and cosines into series whose terms involve sines and cosines of multiple angles.

6. Develop $y = (x + h)^n$.

We have

$$F(x+h) = (x+h)^n; \ F(x) = x^n; \ F'(x) = nx^{n-1};$$
$$F''(x) = n(n-1)x^{n-2}; \text{ etc.}$$

Therefore, by Taylor's formula,

$$F(x+h) = (x+h)^n = x^n + nx^{n-1} h + n\frac{(n-1)}{1.\ 2}x^{n-2} h^2 + \text{etc.},$$

the well known binomial formula.

7. Develop $F(x+h) = \log(x+h)$.

We have

$$F(x+h) = \log(x+h); \ F(x) = \log x; \ F'(x) = \frac{1}{x};$$

$$F''(x) = -\frac{1}{x^2};\ F'''(x) = \frac{1.\ 2}{x^3};\ F^{iv}(x) = -\frac{1.\ 2.\ 3}{x^4};\ \text{etc.}$$

Therefore, by Taylor's formula,

$$F(x+h) = \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4},\ \text{etc.}$$

Corollary 1.—If $x = 1$, then $\log x = 0$, and we have

$$\log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \text{etc.}$$

Corollary 2.—From the last equation, we have, by writing u for h,

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \text{etc.}$$

$$\therefore\ \log\left(1 + \frac{1}{u}\right) = \log(1 + u^{-1}) = u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3} + \text{etc.}$$

$$\therefore\ \log(1+u) - \log\left(1 + \frac{1}{u}\right) = \log\left\{\frac{1+u}{1+\frac{1}{u}}\right\} = \log u$$

$$= (u - u^{-1}) - \tfrac{1}{2}(u^2 - u^{-2}) + \tfrac{1}{3}(u^3 - u^{-3}) - \tfrac{1}{4}(u^4 - u^{-4}) + \text{etc.}$$

Let $u = e^{x\sqrt{-1}}$; whence $\log u = x\sqrt{-1}$, and

$$x\sqrt{-1} = (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) - \tfrac{1}{2}(e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}})$$
$$+ \tfrac{1}{3}(e^{3x\sqrt{-1}} - e^{-3x\sqrt{-1}}) - \text{etc.}$$

$$\therefore\ x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{\sqrt{-1}} - \tfrac{1}{2}\frac{e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}}{\sqrt{-1}}$$
$$+ \tfrac{1}{3}\frac{e^{3x\sqrt{-1}} - e^{-3x\sqrt{-1}}}{\sqrt{-1}},\ \text{etc.}$$

$\therefore$ by Example 4,

$$\frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x, \text{ etc.};$$

and, by differentiation and reduction,

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \cos 4x + \text{etc.}$$

8. By Examples 1, 2, and 4, we have

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x,$$

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x.$$

$\therefore$ by division, $\dfrac{e^{x\sqrt{-1}}}{e^{-x\sqrt{-1}}} = e^{2x\sqrt{-1}} = \dfrac{\cos x + \sqrt{-1}\sin x}{\cos x - \sqrt{-1}\sin x}$

$$= \frac{1 + \sqrt{-1}\operatorname{tang} x}{1 - \sqrt{-1}\operatorname{tang} x}.$$

$\therefore$ passing to logarithms,

$$2x\sqrt{-1} = \log(1 + \sqrt{-1}\operatorname{tang} x) - \log(1 - \sqrt{-1}\operatorname{tang} x) \quad (1).$$

But, $\log(1+u) - \log(1-u) = 2\left(u + \dfrac{u^3}{3} + \dfrac{u^5}{5} + \text{etc.}\right)$,

as may be seen by substituting $\pm u$ for h in Ex. 7, Cor. 1, and subtracting the results.

$\therefore$ $\log(1 + \sqrt{-1}\operatorname{tang} x) - \log(1 - \sqrt{-1}\operatorname{tang} x)$

$$= 2\Big\{\{\sqrt{-1}\operatorname{tang} x\} + \tfrac{1}{3}\{\sqrt{-1}\operatorname{tang} x\}^3$$

$$+ \tfrac{1}{5}\{\sqrt{-1}\operatorname{tang} x\}^5 + \text{etc.}\Big\}.$$

$\therefore$ by substitution in (1),

$$2x\sqrt{-1} = 2\Big\{\{\sqrt{-1}\,\text{tang}\,x\} + \tfrac{1}{3}\{\sqrt{-1}\,\text{tang}\,x\}^3 + \tfrac{1}{5}\{\sqrt{-1}\,\text{tang}\,x\}^5\ \text{etc.}\Big\},$$

$$\text{or,}\ x = \text{tang}\,x - \tfrac{1}{3}\,\text{tang}^3\,x + \tfrac{1}{5}\,\text{tang}^5\,x - \text{etc.}$$

9. Develop $F(x+h) = \sin(x+h)$.

We have

$$F(x+h) = \sin(x+h);\quad F(x) = \sin x;\quad F'(x) = \cos x;$$
$$F''(x) = -\sin x;\quad F'''(x) = -\cos x;\quad F^{\text{iv}}(x) = \sin x;\quad \text{etc.}$$

Therefore, by Taylor's formula,

$$\begin{aligned} F(x+h) &= \sin(x+h) \\ &= \sin x + \frac{h}{1}\cos x - \frac{h^2}{1.\,2}\sin x - \frac{h^3}{1.\,2.\,3}\cos x + \text{etc.} \\ &= \sin x\left\{1 - \frac{h^2}{1.\,2} + \frac{h^4}{1.\,2.\,3.\,4} - \text{etc.}\right\} \\ &\qquad + \cos x\left\{h - \frac{h^3}{1.\,2.\,3} + \text{etc.}\right\} \\ &= \sin x\cos h + \cos x\sin h. \end{aligned}$$

10. Develop $F(x+h) = \cos(x+h)$.

11. Develop $y = \sin^{-1}x$.

We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \\ &= 1 + \frac{x^2}{1.\,2} + \frac{1.\,3}{1.\,2.\,2^2}x^4 + \frac{1.\,3.\,5}{1.\,2.\,3.\,2^3}x^6\ \text{etc.}; \end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{1.\,2.\,x}{1.\,2} + \frac{1.\,3.\,4}{1.\,2.\,2^2}x^3 + \frac{1.\,3.\,5.\,6}{1.\,2.\,3.\,2^3}x^5 + \text{etc.};$$

$$\frac{d^3y}{dx^3} = 1^2 + \frac{1.\,3^2.\,4}{1.\,2.\,\,2^2}x^2 + \frac{1.\,3.\,5^2.\,6}{1.\,2.\,3.\,\,2^3}x^4 + \text{etc.};$$

$$\frac{d^4y}{dx^4} = \frac{1.\,2.\,3^2.\,4}{1.\,2.\,2^2}x + \frac{1.\,3.\,4.\,5^2.\,6}{1.\,2.\,3.\,2^3.}x^3 + \text{etc.};$$

$$\frac{d^5y}{dx^5} = \frac{1^2.\,2.\,3^2.\,4}{1.\,\,2.\,2^2} + \frac{1.\,3^2.\,4.\,5^2.\,6}{1.\,2.\,\,3.\,2^3}x^2 + \text{etc.};$$

Now, making $x = 0$, we have,

$$(y) = 0; \quad \left(\frac{dy}{dx}\right) = 1; \quad \left(\frac{d^2y}{dx^2}\right) = 0; \quad \left(\frac{d^3y}{dx^3}\right) = 1^2;$$

$$\left(\frac{d^4y}{dx^4}\right) = 0; \quad \left(\frac{d^5y}{dx^5}\right) = 1^2.\,3^2; \text{ etc.}$$

$\therefore$ by Maclaurin's formula,

$$y = \sin^{-1}x = x + \frac{1^2.\,x^3}{1.\,2.\,3} + \frac{1^2.\,3^2.\,x^5}{1.\,2.\,3.\,4.\,5} + \text{etc.}$$

12. Develop $y = \text{tang}^{-1}(x)$.

We have

$$\frac{dy}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \text{etc.};$$

$$\frac{d^2y}{dx^2} = -2x + 4x^3 - 6x^5 + 8x^7 - \text{etc.};$$

$$\frac{d^3y}{dx^3} = -2 + 3.\,4x^2 - 5.\,6x^4 + 7.\,8x^6 - \text{etc.};$$

$$\frac{d^4y}{dx^4} = 2.\,3.\,4x - 4.\,5.\,6x^3 + 6.\,7.\,8x^5 - \text{etc.}$$

Making $x = 0$ in these expressions, we have,

$$(y) = 0; \quad \left(\frac{dy}{dx}\right) = 1; \quad \left(\frac{d^2y}{dx^2}\right) = 0; \quad \left(\frac{d^3y}{dx^3}\right) = -2; \text{ etc.}$$

$\therefore$ by Maclaurin's Formula,

$$y = \text{tang}^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}$$

Corollary.—If in this series we make $x = \text{tang } 45° = 1$, we have

$$y = \text{arc of } 45° = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{etc.} = \tfrac{1}{4}\pi; \quad \text{or}$$

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{etc.})$$

This is a slowly converging series for the ratio of the circumference of a circle to its diameter.

Scholium.—In the last two examples we have expanded the values of $\frac{dy}{dx}$, a method which is applicable whenever such expansion will give rise to converging series. When this is not the case, the method ceases to give exact results; for if the series is not converging, we evidently have no right to say that it represents *exactly* the value of the expression with which it is connected by the sign of equality.

In all the preceding examples all of the successive derivatives have proved to be finite, and therefore the Formulas of Taylor and Maclaurin have been applicable. Had this not been the case, it would have been necessary to make use of formula (1) instead of (2) or (3), and (4) instead of (5), (6), or (7).

CHAPTER VI.

THE CONVERGENCE OF SERIES.

54. The formulas developed in the preceding chapter give exact results only when they give rise to converging series: it is consequently important for us to consider the laws of the convergence of series in general.

Let $u_0, u_1, u_2 \ldots\ldots u_n$ be a series, of which u_n is the general term; and let

$$S_n = u_0 + u_1 + u_2 + \ldots u_{n-1}$$

be the sum of the first n terms.

If, as n increases, S_n tends continually toward a *finite limit*, the series is **converging.**

If this is not the case, the series is **diverging.**

Suppose S to be the limit toward which S_n approaches. Then, the sums S_n, S_{n+1}, S_{n+2}, etc., will differ by infinitesimals from S and from each other. Now

$S_{n+1} - S_n = u_n$: $S_{n+2} - S_n = u_n + u_{n+1}$: etc.; and, since these differences are infinitesimals, it follows that when $n = \infty$, we must have

$$u_n = 0; \; u_n + u_{n+1} = 0; \text{ etc.}$$

And, conversely, whenever u_n and all the succeeding terms are *zero* for $n = \infty$, the series tends toward a finite limit, and is therefore converging.

55. These principles furnish us with two simple rules for determining, in many cases, the convergence of series.

RULE 1.—*If we represent by* $\mathrm{U_n}$ *the numerical value of* $\mathrm{u_n}$, *and designate by* L *the value of the limit toward which* $\sqrt[n]{\mathrm{U_n}}$ *approaches as* n *increases, then the series will be converging if* $\mathrm{L} < 1$, *and diverging if* $\mathrm{L} > 1$.

For, let $L < 1$. If we designate by ρ a number less than 1 and greater than L, so that $L < \rho < 1$, and suppose n to increase indefinitely, then

$\sqrt[n]{U_n}$, which differs from L by an infinitesimal, will finally become less than ρ; so that we shall then have

$\sqrt[n]{U_n} < \rho$; and, therefore, $u_n = \pm U_n < \rho^n$, and the terms u_n, u_{n+1}, u_{n+2}, etc., will be less than ρ^n, ρ^{n+1}, ρ^{n+2}, etc.

Now, since ρ is less than 1, the terms of the series, ρ^n, ρ^{n+1}, ρ^{n+2}, etc., evidently diminish as n increases: much more, then, do the terms u_n, u_{n+1}, u_{n+2}, etc., diminish, and therefore the series

$$u_0, \; u_1, \; u_2, \; . \; . \; . \; u_n$$

is in that case converging.

In precisely the same manner, changing the sense of the inequality, may we show that when $L > 1$ the series is diverging.

RULE 2.—*If, as* n *increases,* u_n *decreases, and the ratio* $\frac{U_{n+1}}{U_n}$ *converges toward a limit* l, *the series will be converging when* $l < 1$, *and diverging when* $l > 1$.

Let e be a quantity less than the difference between l and 1, so that the two quantities, $l - e$ and $l + e$, shall be, at the same time with l, less or greater than 1.

Supposing n to increase continually, $\frac{U_{n+1}}{U_n}$ will finally be comprised between $l - e$ and $l + e$, and the terms of the series, u_n, u_{n+1}, u_{n+2}, etc., will be comprised between the corresponding terms of the two series,

$$u_n, \; u_n(l - e), \; u_n(l - e)^2, \text{ etc.,}$$

$$u_n, \; u_n(l + e), \; u_n(l + e)^2, \text{ etc.}$$

Now these series are both converging if $l < 1$, and diverging if $l > 1$. Hence, in the first case, the given series, u_n, u_{n+1}, etc., will be converging, and therefore u_0, u_1, u_2, etc., will also be converging, while in the second case it will be diverging.

Corollary.—The two limits L and l are the same.

For, designating any number by m, the ratios

$$\frac{U_{m+1}}{U_m}, \; \frac{U_{m+2}}{U_{m+1}}, \; \frac{U_{m+3}}{U_{m+2}} \; . \; . \; . \; \frac{U_{m+n}}{U_{m+n-1}},$$

and consequently their geometrical mean, $\sqrt[n]{\dfrac{U_{m+n}}{U_m}}$, will differ from l by a quantity e, whose limit is *zero*. Consequently,

$$\sqrt[n]{U_{m+n}} = \left\{U_{n\left(1+\frac{m}{n}\right)}\right\}^{\frac{1}{n}} = (l \pm e)\left\{U_m\right\}^{\frac{1}{n}}.$$

Passing to the limits, by making $n = \infty$, we have

$$\lim \{U_n\}^{\frac{1}{n}} = L = \lim\left\{(l \pm e)\{U_m\}^{\frac{1}{n}}\right\} = l.$$

56. We will now apply these rules to Taylor's and Maclaurin's formulas.

Let ϕ_n be the value of the expression $\dfrac{1}{1.2\,..\,n} F^n(x)$, and let Φ be the limit of $\sqrt[n]{\phi_n}$ or $\dfrac{\phi_{n+1}}{\phi_n}$. The two formulas will be converging when the numerical value of x is less than $\dfrac{1}{\Phi}$, and diverging in all other cases. For, if we designate the general term of these formulas by $u_n = \phi_n(x^n)$, we shall have, in the first case,

$$\lim (u_n)^{\frac{1}{n}} = \Phi x < 1; \qquad \lim\left(\frac{u_{n+1}}{u_n}\right) = \Phi x < 1;$$

and in the second case,

$$\lim (u_n)^{\frac{1}{n}} = \Phi x > 1; \qquad \lim\left(\frac{u_{n+1}}{u_n}\right) = \Phi x > 1.$$

Therefore, since in the first case the limit is less than 1, the series is converging; while in the second case, the limit being greater than 1, the series is diverging.

57. EXAMPLES.

1. Let $F(x) = e^x$. Is the development of e^x a converging series?

We have

$$F(0)=1; \quad F^n(0)=1; \quad \phi_n=\frac{1}{1.2\ldots n}; \quad \frac{\phi_{n+1}}{\phi_n}=\frac{1}{n+1};$$

$$\Phi=\lim\frac{1}{n+1}=0; \quad \frac{1}{\Phi}=\infty.$$

Every numerical value of x is less than $\frac{1}{\Phi}$, and therefore the development of e^x is converging for every finite value of x.

2. Let $F(x)=\cos x$.

In this case $\Phi=\left(\frac{1}{1.2\ldots n}\right)^{\frac{1}{n}}$. The value of this expression is evidently less than 1, and therefore the development of $\cos x$ is a converging series.

3. Let $F(x)=\sin x$.

The development is a converging series.

4. Let $F(x)=\log(1+x)$.

We have, by differentiation,

$$F^n(0)=(-1)^{n-1}\{1.2.3\ldots n-1\};$$

$$\phi_n=\frac{1}{n}; \quad \frac{\phi_{n+1}}{\phi_n}=\frac{1}{1+\frac{1}{n}}; \quad \Phi=1; \quad \frac{1}{\Phi}=1.$$

Therefore the development of $\log(1+x)$ is not converging for any value of x greater than *unity*.

5. Let $F(x+h)=\log(x+h)$.

Is the development of this function converging?

58. Although the formula of Maclaurin gives, in general, the expression for the development of $F(x)$ whenever it is converging, yet this is not always the case.

If, for example, we take

$$F(x)=e^{-x^2}+e^{-\frac{1}{x^2}}.$$

the development of this function, by Maclaurin's formula, is converging; but instead of being the development of the entire function, it is merely that of the first term, e^{-x^2}.

Nevertheless, whenever the functions represented by $F(x)$, $F(x+h)$, can be developed, by any known process, into converging series, arranged according to the ascending powers of x or h, the resulting series will be identical with those given by Maclaurin's and Taylor's formulas: for two converging series, arranged according to the ascending powers of the same quantity, and whose sums are equal, must, by the theory of indeterminate co-efficients, be equal term for term.

We may remark that it is an open question whether a function which can not be developed by Maclaurin's formula is essentially incapable of development by any process whatever.

CHAPTER VII.

ESTIMATION OF THE VALUES OF FUNCTIONS WHICH ASSUME INDETERMINATE FORMS FOR CERTAIN VALUES OF THE VARIABLES.

59. When the relation between two functions of the same variable is that of a *quotient, product,* or *exponential,* a particular value of the variable may render one or both of the functions equal to *zero* or *infinity,* and thus reduce the given expression to one of the forms, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \times 0$, 0^{∞}, etc. These are usually called **indeterminate** forms, because, being the symbols of operations which it is impossible to effect, they offer, in themselves, no clue by means of which their values may be determined. They are, properly speaking, the *limits* toward which the forms of the expressions from which they originate converge as the value of the variable

tends toward some particular value, say x_0; and the limits toward which the *values* of the given expressions tend, as x tends toward its value x_0, are evidently the **real values** symbolized by the expressions when they assume the peculiar forms above enumerated.

The Calculus affords a simple method of finding these real values in most cases.

60. Functions of the form $\frac{0}{0}$.

Let $F(x)$, $f(x)$ be two functions of x, such that

$$F(x_0) = 0, \quad f(x_0) = 0.$$

Then we shall have

$$\frac{F(x_0)}{f(x_0)} = \frac{0}{0}.$$

We have already seen [Art. 45] that if $F(x)$, $f(x)$ and their first $n-1$ derivatives reduce to *zero* for $x = x_0$, then

$$\frac{F(x_0+h)}{f(x_0+h)} = \frac{F^n(x_0+\theta h)}{f^n(x_0+\theta h)}.$$

$$\text{Hence,} \quad \lim \frac{F(x_0+h)}{f(x_0+h)} = \lim \frac{F^n(x_0+\theta h)}{f^n(x_0+\theta h)}; \quad \text{or,}$$

$$\frac{F(x_0)}{f(x_0)} = \frac{F^n(x_0)}{f^n(x_0)}.$$

Therefore, *The value of a function which reduces to the form $\frac{0}{0}$ for a particular value x_0 of x is equal to the ratio of the first derivatives of the numerator and denominator which do not reduce to 0 or ∞ for the value x_0 of x.*

Corollary.—If either of the derivatives should be 0 or ∞ while the other is finite, the real value of the function will then be 0 or ∞.

Scholium.—If all the derivatives should reduce to 0 or ∞, this method ceases to be applicable. In such cases the proper plan is to substitute x_0+h for x in each of the functions, to find by some algebraic process the resulting value of the fraction, and, finally, to pass to the limit by making $h=0$.

EXAMPLES.

1. Let $\dfrac{x^4-1}{x^3-1}=\dfrac{0}{0}$ for $x=1$.

We have

$$F(x)=x^4-1,\quad f(x)=x^3-1.$$

$$\therefore\ F'(x)=4x^3=4 \text{ and } f'(x)=3x^2=3.$$

$$\therefore\ \frac{F(x)}{f(x)}=\frac{F'(x)}{f'(x)}=\frac{4}{3}.$$

2. $\dfrac{F(x)}{f(x)}=\dfrac{e^x-e^{-x}}{\sin x}=\dfrac{0}{0}$ for $x=0$.

Ans. $\dfrac{F(x)}{f(x)}=\dfrac{F'(x)}{f'(x)}=\dfrac{2}{1}=2$, for $x=0$.

3. $\dfrac{F(x)}{f(x)}=\dfrac{\sin x}{x^2}=\dfrac{0}{0}$ for $x=0$.

Ans. $\dfrac{F(x)}{f(x)}=\dfrac{F'(x)}{f'(x)}=\dfrac{\cos x}{2x}=\dfrac{1}{0}=\infty$, for $x=0$.

4. $\dfrac{F(x)}{f(x)}=\dfrac{\log x}{x-1}=\dfrac{0}{0}$, for $x=1$.

Ans. $\dfrac{F(x)}{f(x)}=\dfrac{F'(x)}{f'(x)}=\dfrac{1}{x}=1$, for $x=1$.

5. $\dfrac{Fx}{f(x)}=\dfrac{a^x-b^x}{x}=\dfrac{0}{0}$, for $x=0$.

Ans. $\dfrac{F(x)}{f(x)}=\log\left(\dfrac{a}{b}\right)$.

6. $\dfrac{F(x)}{f(x)}=\dfrac{x^4-5x^3+\ \ 9x^2-\ \ 7x+2}{x^4-6x^3+12x^2-10x+3}=\dfrac{0}{0}$, for $x=1$.

We have

$$\frac{F'(x)}{f'(x)}=\frac{4x^3-15x^2+18x-\ 7}{4x^3-18x^2+24x-10}=\frac{0}{0};$$

$$\frac{F''(x)}{f''(x)}=\frac{12x^2-30x+18}{12x^2-36x+24}=\frac{0}{0};$$

$$\frac{F'''(x)}{f'''(x)}=\frac{24x-30}{24x-36}=\frac{1}{2},\text{ for } x=1.$$

$$\therefore\ \frac{F(x)}{f(x)}=\tfrac{1}{2},\text{ for } x=1.$$

7. $\dfrac{F(x)}{f(x)}=\dfrac{\text{tang } x-\sin x}{\sin^3 x}=\dfrac{0}{0}$, for $x=0$.

Ans. $\frac{1}{2}$, for $x=0$.

8. $\dfrac{F(x)}{f(x)}=\dfrac{\sqrt[3]{x-a}}{\sqrt[4]{x^2-a^2}}=\dfrac{0}{0}$, for $x=a$.

In this example all the derivatives become infinite, for $x=a$. We therefore place $x=a+h$, and after reducing the resulting expression as much as possible, put $h=0$.

We thus have,

$$\frac{\sqrt[3]{x-a}}{\sqrt[4]{x^2-a^2}}=\frac{\sqrt[3]{h}}{\sqrt[4]{2ah+h^2}}=\frac{h^{\frac{1}{3}}}{h^{\frac{1}{4}}(2a+h)^{\frac{1}{4}}}=\frac{h^{\frac{1}{12}}}{(2a+h)^{\frac{1}{4}}}=0,$$

for $x=a$, and $h=0$.

9. $\dfrac{F(x)}{f(x)}=\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\dfrac{0}{0}$, for $x=a$.

Placing $x=a+h$, since the derivatives are all infinite, we have,

$$\frac{(a+h)^{\frac{1}{2}}-a^{\frac{1}{2}}+h^{\frac{1}{2}}}{(2ah+h^2)^{\frac{1}{2}}}=\frac{h^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}h, \text{ etc.}}{h^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}}=\frac{1+\text{etc.}}{(2a+h)^{\frac{1}{2}}}$$

$$=\frac{1}{(2a+h)^{\frac{1}{2}}}=\frac{1}{(2a)^{\frac{1}{2}}}, \text{ for } x=a, \text{ and } h=0.$$

10. $\dfrac{F(x)}{f(x)}=\dfrac{e^x-1-\log(1+x)}{x^2}=\dfrac{0}{0}$, for $x=0$.

Instead of differentiating as usual, we shall evaluate this expression by substituting for e^x and $\log(1+x)$ their expanded values. These are

$$e^x=1+x+\frac{x^2}{1.\,2}+\frac{x^3}{1.\,2.\,3}+\text{etc.}$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\text{etc.}$$

$$\therefore \quad \frac{F(x)}{f(x)}=1-\frac{x}{6}, \text{ etc.}, =1 \text{ when } x=0.$$

61. Functions of the form $\frac{\infty}{\infty}$.

If $\dfrac{F(x)}{f(x)}=\dfrac{\infty}{\infty}$ for any particular value of x, then $\dfrac{1}{F(x)}$ and $\dfrac{1}{f(x)}$ will each be zero for the same value of x, and we may solve this case by applying the preceding method to the functions $\dfrac{1}{F(x)}, \dfrac{1}{f(x)}$. We have, therefore,

$$\frac{F(x_0)}{f(x_0)}=\frac{\infty}{\infty}=\frac{\dfrac{1}{f(x_0)}}{\dfrac{1}{F(x_0)}}=\frac{0}{0}.$$

Hence, by differentiation,

$$\frac{F(x_0)}{f(x_0)}=\frac{\dfrac{f'(x_0)}{\{f(x_0)\}^2}}{\dfrac{F'(x_0)}{\{F(x_0)\}^2}}=\left\{\frac{F(x_0)}{f(x_0)}\right\}^2 \cdot \frac{f'(x_0)}{F'(x_0)}.$$

Whence, by reduction,

$$\frac{F(x_0)}{f(x_0)}=\frac{F'(x_0)}{f'(x_0)}=\;\cdots\cdots\;\frac{F^n(x_0)}{f^n(x_0)};$$

and the method in this case is therefore the same as for the form $\frac{0}{0}$. It is to be observed, however, that if $F(x)$ is infinite for a finite value of x, so also are all of its derivatives; and the above method will not be of practical value for finite values of x, unless we can detect in the terms of the derived functions a common factor, which, being canceled, will leave a result whose value can be determined.

Examples.

1. $$\frac{F(x)}{f(x)}=\frac{x^n}{e^x}=\frac{\infty}{\infty},\text{ for } x=\infty.$$

Differentiating n times, x disappears from the numerator, and we have,

$$\frac{F(x)}{f(x)}=\frac{n(n-1)(n-2)\;\ldots\;3.\,2.\,1}{e^x}=0,\text{ for } x=\infty.$$

2. $$\frac{F(x)}{f(x)}=\frac{\log(x)}{\left(\frac{1}{x}\right)}=-\frac{\infty}{\infty},\text{ for } x=0.$$

We have,

$$\frac{F(x)}{f(x)}=\frac{F'(x)}{f'(x)}=\frac{\frac{1}{x}}{-\frac{1}{x^2}}=-x=0,\text{ for } x=0.$$

62. Functions of the form $\infty\times 0$.

Let $F(x)$, $f(x)$, be two functions of x, such that for $x=x_0$ we shall have $F(x_0)=\infty$, $f(x_0)=0$.

Then we shall have, identically,

$$F(x_0) \times f(x_0) = \frac{f(x_0)}{\frac{1}{F(x_0)}} = \frac{0}{0},$$

which reduces this case to the first.

Therefore, by differentiation,

$$F(x_0) \times f(x_0) = -\{Fx_0\}^2 \times \frac{f'(x_0)}{F'(x_0)}.$$

If the second number of this equation should not prove to be determinable, we must continue to differentiate as before, and it is obvious that the method will have the same limitations as in the preceding case.

63. Functions of the forms 0^0, ∞^0, 0^∞, etc.

Let $u = F(x)^{f(x)}$, in which either or both of the functions $F(x)$, $f(x)$, may reduce to 0 or ∞, for the value x_0 of x.

Passing to logarithms, we have,

$$\log u = f(x) \log F(x) = 0 \times \infty.$$

We may, therefore, find the value of $\log u$, or of

$$f(x) \log F(x)$$

by one of the preceding methods, and we shall have, finally,

$$u = e^{\log u}.$$

EXAMPLES.

1. $$F(x)^{f(x)} = x^x = 0^0, \text{ for } x = 0.$$

Passing to logarithms, we have,

$$\log x^x = x \log x = -0 \times \infty, \text{ for } x = 0.$$

$$\therefore \quad \log x^x = x \log x = \frac{\log x}{\left(\frac{1}{x}\right)} = -\frac{\infty}{\infty} = 0,$$

as in Ex. 2, Art. 61.

$$\therefore \quad x^x = e^{\log x^x} = e^0 = 1, \text{ for } x = 0.$$

2. $$F(x)^{f(x)} = (1 + x)^{\frac{1}{x}} = 1^\infty, \text{ for } x = 0.$$

We have

$$\log(1 + x)^{\frac{1}{x}} = \frac{1}{x} \log(1 + x) = \frac{0}{0}, \text{ for } x = 0.$$

$\therefore$ by differentiation,

$$\log(1 + x)^{\frac{1}{x}} = \frac{1}{1 + x} = 1, \text{ for } x = 0.$$

$$\therefore \quad (1 + x)^{\frac{1}{x}} = e, \text{ for } x = 0.$$

3. $$F(x)^{f(x)} = x^{\frac{1}{x}} = \infty^0, \text{ for } x = \infty.$$

We have

$$\log x^{\frac{1}{x}} = \frac{1}{x} \log x = \frac{\infty}{\infty}, \text{ for } x = \infty.$$

$\therefore$ by differentiation,

$$\log x^{\frac{1}{x}} = \frac{1}{x} = 0, \text{ and } x^{\frac{1}{x}} = 1, \text{ for } x = \infty.$$

64. General Examples on this Chapter.

1. $\dfrac{(2a^3x - x^4)^{\frac{1}{2}} - a(a^2x)^{\frac{1}{3}}}{a - (ax^3)^{\frac{1}{4}}} = \dfrac{0}{0}$ when $x = a$. Ans. $\dfrac{16a}{9}$.

2. $\dfrac{x^m - x^{m+n}}{1-x^{2p}} = \dfrac{0}{0}$ when $x=1$. Ans. $\dfrac{n}{2p}$.

This expression may be put under the form $\dfrac{x^m}{1+x^p} \cdot \dfrac{1-x^n}{1-x^p}$, and since the latter factor alone reduces to $\dfrac{0}{0}$, we may find its real value when $x=1$, and multiply the result by the first factor, whose value is evidently $\dfrac{1}{2}$.

3. $\dfrac{1-\sin x + \cos x}{\sin x + \cos x - 1} = \dfrac{0}{0}$ when $x = \dfrac{\pi}{2}$. Ans. 1.

4. $\dfrac{(x^2 - a^2)\sin\dfrac{\pi x}{2a}}{x^2 \cos\dfrac{\pi x}{2a}} = \dfrac{0}{0}$ when $x = a$. Ans. $-\dfrac{4}{\pi}$.

In this example $\sin\dfrac{\pi x}{2a}$ reduces to 1, and we may proceed as in Ex. 2.

5. $\dfrac{\cos^{-1}(1-x)}{\sqrt{2x - x^2}} = \dfrac{0}{0}$ when $x=0$. Ans. 1.

6. $\dfrac{\text{tang}\ \pi x - \pi x}{2x^2\ \text{tang}\ \pi x} = \dfrac{0}{0}$ when $x=0$.

Assuming $x = 0 + h$, and expanding $\text{tang}\ \pi x = \text{tang}\ \pi h$, by Maclaurin's formula, we have,

$$\text{tang}\ \pi h = \pi h + \frac{(\pi h)^3}{3} + \text{etc.}$$

$$\therefore \frac{\text{tang}\ \pi h - \pi h}{2h^2\ \text{tang}\ \pi h} = \frac{1}{2h^2} - \frac{\pi}{2h(\pi h + \frac{1}{3}(\pi h)^3 + \text{etc.})}$$

$$= \frac{\pi + \frac{1}{3}\pi^3 h^2 + \text{etc.} - \pi}{2h(\pi h + \frac{1}{3}(\pi h)^3 + \text{etc.})} = \frac{\pi^2}{6} \text{ when } h=0, \text{ or } x=0.$$

7. $\left(1+\dfrac{1}{x}\right)^x=\infty^0$ when $x=0$. Ans. 1.

8. $\left(\dfrac{\text{tang } x}{x}\right)^{\frac{1}{x}}=1^\infty$ when $x=0$. Ans. 1.

9. $$\frac{1-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1+x^2}}=\frac{0}{0} \text{ when } x=0.$$

10. $$(\sin x)^{\text{tang } x}=1^\infty \text{ when } x=\frac{\pi}{2}.$$

11. $$\frac{\log \text{tang } 2x}{\log \text{tang } x}=\frac{\infty}{\infty} \text{ when } x=0.$$

12. $$\left(\frac{1}{x^n}\right)^{\frac{m}{x}}=\infty^0 \text{ when } x=0.$$

13. $$\frac{x}{x-1}-\frac{1}{\log x}=\infty-\infty \text{ when } x=1.$$

This may be put under the form

$$\frac{x\log x-(x-1)}{(x-1)\log x}=\frac{0}{0} \text{ when } x=1. \qquad \text{Ans. } \frac{1}{2}.$$

14. $(x)^{\frac{1}{1-x}}=1^\infty$ when $x=1$. Ans. $\dfrac{1}{e}$.

15. $(Ax^m+Bx^{m-1}+ \cdot\cdot\cdot\; Mx+N)^{\frac{1}{x}}=\infty^0$ when $x=\infty$.
Ans. 1.

16. $(\cos ax)^{\text{cosec}^2 bx}=1^\infty$ when $x=0$. Ans. $\dfrac{a^2}{e^{2b^2}}$.

17. Show that when $x=\infty$,

$$\frac{a^{x^m}}{b^{x^n}}=\infty \text{ or } 0, \text{ according as } m > \text{ or } < n;$$

a and b being both greater than unity.

CHAPTER VIII.

MAXIMA AND MINIMA.

65. When for a particular value x_0 of x the corresponding value $F(x_0)$ of $F(x)$ is greater than the values $F(x_0+h)$ and $F(x_0-h)$, in which h is an infinitesimal, the value $F(x_0)$ is said to be a **maximum** value of $F(x)$.

If $F(x_0)$ be less than $F(x_0+h)$ and $F(x_0-h)$, then $F(x_0)$ is said to be a **minimum** value of $F(x)$.

It results from these definitions that, *for a maximum*, $F(x_0+h)-F(x_0)$ must be *negative*, whatever may be the sign of h; and that, in order to a *minimum*, $F(x_0+h)-F(x_0)$ must be *positive*, whatever may be the sign of h.

[1] It is evident that in order that these conditions may be fulfilled, $F(x)$ must be a *decreasing* function on one side of $F(x_0)$, and an *increasing* function on the other side of $F(x_0)$. Now, we have already seen [Art. 18] that when $F(x)$ is decreasing, $F'(x)$ is *negative;* and when $F(x)$ is increasing, $F'(x)$ is *positive.*

Hence, in order that $F(x)$ may be a maximum or minimum, $F'(x)$ must, in passing through the particular value $F'(x_0)$, change from a state of increase to one of decrease, or *vice versâ*: and, as a quantity can change its sign only in passing through 0 or ∞, we have, as a necessary condition for a maximum or minimum value of $F(x)$,

$$F'(x_0)=0, \text{ or } F'(x_0)=\infty.$$

[2] Now, it has already been shown [Art. 46] that

$$F(x_0+h)-F(x_0)=\frac{h^n}{1.2\,..\,n}F^n(x_0+\theta h),$$

an equation which may be written

$$F(x_0+h)-F(x_0)=\frac{h^n}{1.2\,..\,n}\{F^n(x_0)+R_nh\},$$

in which n denotes the order of the first derivative which is *finite* for the particular value x_0 of x, and $R_n h$ is an infinitesimal.

The sign of the second member of this equation will evidently depend upon that of its first term $\frac{h^n}{1.2\,..\,n} F^n(x_0)$.

Hence,

If n be an *odd number*, this term and, therefore, the first member of the equation will change sign with h, and $F(x_0)$ can not be either a maximum or minimum.

If n be an *even number*, the second member and, therefore, the first member also will *not* change sign with h: it will be *positive* if $F^n(x_0)$ is greater than zero, and *negative* if $F^n(x_0)$ is less than zero.

But $F(x_0)$ is a maximum if $F(x_0+h) - F(x_0) < 0$, and a minimum if $F(x_0+h) - F(x_0) > 0$. Therefore,

$F(x_0)$ is a *maximum* if $F^n(x_0) < 0$; and

$F(x_0)$ is a *minimum* if $F^n(x_0) > 0$.

66. The preceding investigation leads to the following **Rule** *for determining maxima and minima values of explicit functions of a single variable.*

1*st.* Form the successive derivatives of the function.

2*d.* Place the first derivative equal to zero or infinity, and solve the resulting equation. The values of x so found are the only values for which the function is to be examined.

3*d.* Substitute these values of x in the remaining derivatives. If the first derivative whose resulting value is *finite* be of an *even* order, and *less than zero*, the corresponding value of $F(x)$ is a *maximum*; but if it be *greater than zero*, the corresponding value of $F(x)$ is a *minimum.*

If the first *finite* derivative be of an *odd* order, there is neither maximum nor minimum.

67. If ALL of the derivatives be either *zero* or *infinity,* the preceding method ceases to apply. In such cases the question may be determined in the following manner:

Substitute $x_0 + h$ for x in the function. Then, if the term which contains the lowest power of h changes sign with h, there is evidently neither a maximum nor minimum value of $F(x)$ for the value x_0 of x. But if this term does not change sign with h, then the value of $F(x_0)$ will be a *maximum* when this term is *negative,* and a *minimum* when it is *positive:* since, in the first case, $F(x_0)$ will be greater than the immediately preceding and following values of $F(x)$, and in the second case it will be less than those values.

It is possible, and sometimes preferable, to use this method even in cases where the general rule is applicable.

68. If it be required to determine **maxima** and **minima** values of **implicit functions** of a single variable, we may, according to the rules established for these functions, determine the successive derivatives, and then proceed as above.

Thus, if we have,

$$u = F(x,y) = 0, \quad \text{then } \frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} \quad (1).$$

But for a maximum or minimum $\frac{dy}{dx} = 0$, and, therefore, $\frac{du}{dx} = 0$, or $\frac{du}{dy} = \infty$. Either of these equations, together with the given equation, $F(x,y) = 0$, will enable us to determine values of x and y for which the function is to be examined, and we may then apply the usual tests to the successive derivatives formed from equation (1).

69. In applying the preceding methods, we may be greatly facilitated by bearing in mind the following simple and self-evident principles:

1st. If $F(x)$ is a maximum or minimum, $a\,F(x)$ is also a maximum or minimum. Therefore, the given function may be multiplied or divided by any *constant factor* without affecting the character of the result.

2d. If $F(x)$ is a maximum or minimum, $F(x) \pm a$ will be a maximum or minimum; but $a - F(x)$ will be a minimum when $F(x)$ is a maximum, and a maximum when $F(x)$ is a minimum.

3d. If $F(x)$ is a maximum or minimum, $\frac{1}{F(x)}$ will be a minimum or maximum.

4th. If $F(x)$ is a maximum or minimum and *positive*, $\{F(x)\}^n$ will also be a maximum or minimum, n being *positive.* But if $F(x)$ be *negative,* $\{F(x)\}^{2n}$ will be a maximum when $F(x)$ is a minimum, and a minimum when $F(x)$ is a maximum.

5th. If $F(x)$ is a maximum or minimum, $\log\{F(x)\}$ will be the same.

6th. It is not admissible to assume x equal to *infinity* in the search for maxima and minima, for in that case x can not have a succeeding value.

7th. The basis of the whole theory is that $F(x)$ must be *continuous,* at least in the neighborhood of the particular values to be examined.

70. EXAMPLES.

1. Divide a number a into two parts whose product shall be a maximum.

Let x be one of the parts, and $a - x$ the other.

$$\text{Then, } F(x) = x(a-x) = ax - x^2.$$

$$F'(x) = a - 2x = 0. \quad \therefore x = \frac{a}{2}.$$

$$F''(x) = -2.$$

The first derivative which does not reduce to zero being of an even order, it follows that the value $\frac{a}{2}$ of x, which reduces the *first* derivative to zero, will render the value of $F(x)$ a maximum. This value is $\frac{a^2}{4}$.

2. Find the value of x which shall render

$$F(x) = x^2 - 8x + 5,$$

a maximum or minimum.

We have,
$$F'(x) = 2x - 8 = 0. \quad \therefore \ x = 4.$$
$$F''(x) = +2.$$

The *second* derivative being positive, $x = 4$ renders $F(x)$ a minimum. The minimum value of $F(x)$ is -11.

3. Find the values of x which shall render

$$F(x) = 2x^3 - 9ax^2 + 12a^2x - 4a^3,$$

a maximum or minimum.

We have,

$$F'(x) = 6x^2 - 18ax + 12a^2 = 0. \quad \therefore \ x = a \text{ and } x = 2a.$$
$$F''(x) = 12x - 18a = -6a, \text{ for } x = a,$$
$$= +6a, \text{ for } x = 2a.$$

Therefore $x = a$ renders $F(x)$ a maximum, and $x = 2a$ renders it a minimum. These values are, respectively, a^3 and 0.

4. Find whether $F(x) = b + c(x - a)^{\frac{7}{3}}$ has maximum or minimum values

We have,

$$F'(x) = \frac{7}{3}c(x-a)^{\frac{4}{3}} = 0. \quad \therefore \ x = a.$$

$$F''(x) = \frac{28}{9}c(x-a)^{\frac{1}{3}} = 0, \quad \text{for } x = a,$$

and all succeeding derivatives are infinite. We infer that the given function has neither maximum nor minimum values.

5. Determine the maximum and minimum values of

$$F(x) = b + c(x - a)^{\frac{2}{3}}.$$

We have,

$$F'(x) = \frac{2}{3}c(x-a)^{-\frac{1}{3}} = \infty, \text{ for } x = a.$$

$$F''(x) = -\frac{2}{9}c(x-a)^{-\frac{4}{3}} = \infty, \text{ for } x = a,$$

and all the succeeding derivatives are infinite.

Let $x = a + h$.

Then, $F(x) = b + ch^{\frac{2}{3}}$.

As this does not change sign with h, and the term containing h is positive, we infer that $x = a$ renders $F(x) = b$ a minimum.

If c be negative, $F(x) = b$ will be a maximum.

6. Find the maximum and minimum values of

$$F(x) = m \sin(x - a) \cos x.$$

We have,

$$F'(x) = m\cos(2x - a) = 0. \quad \therefore x = \frac{a}{2} \pm \frac{\pi}{4}.$$

$$F''(x) = -2m \sin(2x - a)$$

$$= -2m, \text{ for } x = \frac{a}{2} + \frac{\pi}{4},$$

$$= +2m, \text{ for } x = \frac{a}{2} - \frac{\pi}{4}.$$

Therefore, the first value of x renders $F(x)$ a maximum, and the second renders $F(x)$ a minimum.

These values of $F(x)$ are, respectively,

$$\frac{m}{2}(1-\sin a) \text{ and } -\frac{m}{2}(1+\sin a).$$

7. Find the maximum and minimum values of y in the expression

$$F(x,y)=y^3-3a^2x+x^3=0.$$

We find,

$$\frac{dy}{dx}=\frac{a^2-x^2}{y^2}=0. \quad \therefore\ x=\pm a.$$

$$\frac{d^2y}{dx^2}=-\frac{2a}{y^2}, \text{ for } x=+a.$$

$$=+\frac{2a}{y^2}, \text{ for } x=-a.$$

$\therefore\ x=+a$ renders y a maximum, and $x=-a$ renders y a minimum.

8. Find the greatest and least ordinates of the curve

$$a^2y-ax^2+x^3=0.$$

We find,

$$\frac{dy}{dx}=\frac{2ax-3x^2}{a^2}=0. \quad \therefore\ x=0 \text{ or } \frac{2a}{3}.$$

$$\frac{d^2y}{dx^2}=\frac{2a-6x}{a^2}=\quad\frac{2}{a}, \text{ for } x=0,$$

$$=-\frac{2}{a}, \text{ for } x=\frac{2a}{3}.$$

$\therefore\ x=0$ renders $y=0$, a minimum; and $x=\frac{2a}{3}$ renders $y=\frac{4a}{27}$, a maximum.

9. Find the number of equal parts into which a number a must be divided in order that their product may be a maximum.

Let x = number of parts, and $\frac{a}{x}$ the value of each part.

Then $\frac{a}{x}\cdot\frac{a}{x}\cdot\frac{a}{x}\ \ldots=\left(\frac{a}{x}\right)^x=$ the required product.

Assume $F(x)=\left(\frac{a}{x}\right)^x$, or by taking the logarithm,

$$F(x)=x\log\left(\frac{a}{x}\right)=x\log a-x\log x.$$

$$\therefore\quad F'(x)=\log a-\log x-1=0;$$

$$\therefore\quad \log a-\log x=\log\left(\frac{a}{x}\right)=1.$$

$$F''(x)=-\frac{1}{x}.$$

Hence, $\qquad \log\left(\frac{a}{x}\right)=1$, or $\frac{a}{x}=e$, and $x=\frac{a}{e}\cdot$

The maximum product is $e^{\frac{a}{e}}$.

10. Find the value of x which renders $F(x)=\frac{x^2}{(a^2+x^2)^3}$ a maximum.

We have,

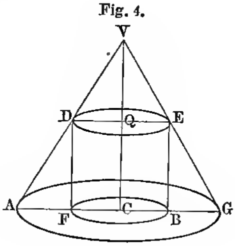
$$F'(x)=\frac{2x(a^2-2x^2)}{(a^2+x^2)^4}=0.\quad\therefore\ x=0\ \text{or}\ \frac{a}{\sqrt{2}}\cdot$$

The value $x=\frac{a}{\sqrt{2}}$ renders $F(x)=\frac{4}{27a^4}$ a maximum.

NOTE.—Whenever we are certain, from the nature of the problem, that there *must* be a maximum or minimum, we may neglect the second derivative, unless the value of x, which reduces the first to zero or infinity, should also reduce the second to the same values. This can often be determined by simple inspection.

11. Find the greatest cylinder which can be inscribed in a given right cone with a circular base.

Fig. 4.



In Fig. 4, let $AC = b$, $VC = a$, $DF = x$.

Then we have,

$$VC : VQ = AC : DQ$$

$$= \frac{AC \times VQ}{VC} = \frac{b(a-x)}{a}:$$

and the volume of the cylinder is

$$\frac{\pi(a-x)^2 b^2 x}{a^2} \quad (1).$$

Therefore, omitting constant factors,

$$F(x) = x(a-x)^2.$$

$$F'(x) = a^2 - 4ax + 3x^2 = 0. \quad \therefore x = a, \text{ or } \frac{a}{3}.$$

We see at once that the second value of x is the one required. This value substituted in (1) will give the volume of the maximum cylinder.

12. The content of a right cone being given, find its form when its surface is a maximum.

Let $\frac{\pi a^3}{3}$ be the given content, x the radius of the base, and y the altitude of the cone.

Then we shall have,

$$\frac{\pi x^2 y}{3} = \frac{\pi a^3}{3}, \text{ whence } x^2 y = a^3, \text{ and } y = \frac{a^3}{x^2}.$$

The area of the base will be πx^2, and the convex surface $\pi x \sqrt{x^2 + y^2}$. Hence,

$$\text{the entire surface} = \pi x^2 + \pi x \sqrt{x^2 + y^2}$$

$$= \pi x^2 + \frac{\pi}{x}\sqrt{x^6 + a^6}.$$

$\therefore$ omitting the constant factor π,

$$F(x) = x^2 + \frac{1}{x}(x^6 + a^6)^{\frac{1}{2}}.$$

We find, $x = \frac{a}{\sqrt{2}}$ and $y = 2a$.

These two elements determine the form of the cone.

13. Find the point on the straight line, joining two lights of unequal intensity, which receives the *least* amount of illumination from them both.

Let a and b denote the intensities of the lights at a unit's distance, designate by c the distance between the two lights, and let x be the distance of the required point from the first light.

Then, since the intensity of light varies inversely as the square of the distance, that of the first light at the distance x will be $\frac{a}{x^2}$, and that of the second will be $\frac{b}{(c-x)^2}$, at the distance $c - x$. Hence,

$$F(x) = \frac{a}{x^2} + \frac{b}{(c-x)^2}.$$

$$F'(x) = -\frac{2a}{x^3} + \frac{2b}{(c-x)^3} = \frac{2bx^3 - 2a(c-x)^3}{x^3(c-x)^3} = 0.$$

$$\therefore \frac{x^3}{(c-x)^3} = \frac{a}{b} : \frac{x}{c-x} = \sqrt[3]{\frac{a}{b}}; \quad x = \frac{c\sqrt[3]{a}}{\sqrt[3]{a} + \sqrt[3]{b}}.$$

14. Find the greatest rectangle which can be cut from a given trapezoid.

15. Find the greatest cylinder which can be cut from a given frustum of a cone.

16. Divide a into two such parts that the product of the m^{th} power of the first by the n^{th} power of the second shall be a maximum.

We have,

$$F(x) = x^m (a - x)^n.$$

$$F'(x) = x^{m-1}(a-x)^{n-1}\{ma - (m+n)x\} = 0.$$

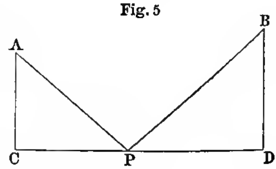
$$\therefore \quad x = 0, \quad x = a, \quad x = \frac{ma}{m+n}.$$

$$F''(x) = \{ma - (m+n)x\}\frac{d\{x^{m-1}(a-x)^{n-1}\}}{dx}$$
$$- (m+n)x^{m-1}(a-x)^{n-1};$$

and by substituting the values of x in this expression, we find that $x = 0$ renders $F(x)$ a minimum if m be *even*, $x = a$ renders $F(x)$ a minimum if n be even, and $x = \frac{ma}{m+n}$ renders $F(x)$ a maximum without reference to the characters of m and n.

17. From two points, A and B, draw lines, AP, BP, to a given line, CD, so that their sum may be a minimum.

Fig. 5



Take the given line as the

axis of abscissas, draw AC and BD perpendicular to CD, and take C as the origin.

Let $CP = x$, $CA = a$, $CD = b$, $BD = d$, $PD = b - x$.

Then we shall have

$$F(x) = \sqrt{a^2 + x^2} + \sqrt{d^2 + (b - x)^2}.$$

$$F'(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{b - x}{\sqrt{d^2 + (b - x)^2}} = 0;$$

$$\therefore \quad \frac{x}{\sqrt{a^2 + x^2}} = \frac{b - x}{\sqrt{d^2 + (b - x)^2}}.$$

The first member of this equation is the cosine of APC, and the second member is the cosine of BPD. Whence, $APC = BPD$, which defines the position of P.

18. Given the length of the arc of a circle; find the angle which it must subtend at the center in order that the corresponding segment may be a maximum.

Let a denote half the length of the arc, and let x be the radius; then $\frac{a}{x}$ is half the angle of the segment and we shall find,

$$\text{Area of segment} = F(x) = ax - x^2 \sin\frac{a}{x} \cos\frac{a}{x}$$

$$\therefore \ F'(x) = a - \left\{ 2x \sin\frac{a}{x} \cos\frac{a}{x} + x^2 \cos\frac{a}{x} \frac{d}{dx} \sin\frac{a}{x} \right.$$

$$\left. + x^2 \sin\frac{a}{x} \frac{d}{dx} \cos\frac{a}{x} \right\} = 0.$$

$\therefore$ by reduction,

$$\cos\frac{a}{x} \left\{ a \cos\frac{a}{x} - x \sin\frac{a}{x} \right\} = 0, \text{ or } \cos\frac{a}{x} = 0.$$

$\therefore\ \dfrac{a}{x} = \dfrac{\pi}{2}$, and $\dfrac{2a}{x} = \pi$, or the required angle is $180°$.

19. Show that of all circular sectors of the same perimeter the one which has the greatest area is that whose arc is double the radius.

20. Find the sides of the greatest rectangle which can be circumscribed about a given rectangle whose sides are a and b.

Ans. Each side $= \dfrac{a+b}{\sqrt{2}}$.

21. Find the sides of the maximum rectangle which can be inscribed in a given ellipse.

22. Find the least ellipse which can be circumscribed about a given parallelogram.

Let $2a$ and $2b$ be the sides of the parallelogram, and designate by α their included angle.

Then the ellipse will have a pair of conjugate diameters parallel to the sides of the parallelogram, and designating them by $2A'$, $2B'$, we shall have,

$$A'^2 a^2 + B'^2 b^2 = A'^2 B'^2 \qquad (1).$$

Also, designating the semi-axes of the ellipse by A and B, we have $A'B' \sin \alpha = AB$, and the area

$$\pi AB = \pi A'B' \sin \alpha \qquad (2).$$

Taking A' as the independent variable, we have

$$F(A') = \pi A'B' \sin \alpha = A'B'.$$

Substituting in this equation the value of B' derived from (1), and differentiating, we shall find that

Area of ellipse : area of parallelogram $= \pi : 2$.

23. Find the length of the longest line that can be drawn from the focus of a hyperbola perpendicular to a tangent.

NOTE.—The formula for the length of a perpendicular from the focus to a tangent is $b\sqrt{\frac{a+ex}{a-ex}}$, in which x is the abscissa of the point of tangency.

24. Two focal chords are drawn in an ellipse at right angles to each other: find their position with reference to the major axis when their sum is a maximum.

25. Two vessels are sailing at right angles to each other, with velocities a and b per hour. Show that if at any given instant their distances from the point of intersection of their courses are p and q, respectively, then their least distance from each other is $\frac{aq-bp}{\sqrt{a^2+b^2}}$.

CHAPTER IX.

DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES.

71. It has been shown in Art. 34 that if we have

$$u = F(x,y),$$

in which equation x and y may be either mutually dependent or entirely independent of each other, then we shall also have

$$\Delta u = \frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y + \alpha \quad (1).$$

In this equation Δx and Δy are the arbitrary but infinitesimal increments of x and y; α is an infinitesimal with reference to Δx and Δy; $\frac{du}{dx}$, $\frac{du}{dy}$, are the partial derivatives of u with respect to x and y; and Δu is the total increment produced in u by the increments assigned to x and y.

Now we have seen [Art. 15, *et seq.*] that whenever infinitesimals occur in the terms of a series or ratio whose limit is to be taken, they may be replaced by others which differ from them by infinitesimals of higher order; and we have also seen [Art. 32] that the differentials of variables may be considered as differing from their infinitesimal differences by infinitesimals. Hence, if we take the limits of the quantities in equation (1), we may write

$\frac{du}{dx}dx + \frac{du}{dy}dy$ for the second member, and du for the first member, thus giving

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \quad (2)$$

Now as Δu is the *total increment* produced in u by a change in x and y, so we may call du **the total differential** of u, while the terms $\frac{du}{dx}dx$, $\frac{du}{dy}dy$, are the **partial differentials** of u with respect to x and y.

We have, therefore, the following

Rule.—The total differential of an explicit function of two (or more) variables is equal to the sum of its partial differentials.

72. Problem.—To find the successive differentials of

$$u = F(x,y).$$

We have

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \quad (1).$$

Differentiating this equation, and observing that $\frac{du}{dx}$, $\frac{du}{dy}$, are themselves functions of both x and y, we have

$$d^2 u = \frac{d\left\{\frac{du}{dx}dx\right\}}{dx}dx + \frac{d\left\{\frac{du}{dx}dx\right\}}{dy}dy + \frac{d\left\{\frac{du}{dy}dy\right\}}{dx}dx$$

$$+ \frac{d\left\{\frac{du}{dy}dy\right\}}{dy}dy.$$

$$\text{Or,}\quad d^2u = \frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy\,dx}dy\,dx + \frac{d^2u}{dy^2}dy^2 \quad (2).$$

This expression may be simplified by observing that

$$\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx};$$

for, designating by Δ_x the increment which a function receives in virtue of the increment Δx to x, we shall have

$$\Delta_x\left\{\frac{du}{dy}\right\} = d\left\{\frac{u+\Delta_x u}{dy}\right\} - \frac{du}{dy} = d\left\{\frac{\Delta_x u}{dy}\right\}.$$

Dividing by Δx, which may be regarded as constant,

$$\frac{\Delta_x\left\{\frac{du}{dy}\right\}}{\Delta x} = \frac{d\left\{\frac{\Delta_x u}{dy}\right\}}{\Delta x} = \frac{d\left\{\frac{\Delta_x u}{\Delta x}\right\}}{dy}.$$

and passing to the limit,

$$\frac{d\left\{\frac{du}{dy}\right\}}{dx} = \frac{d\left\{\frac{du}{dx}\right\}}{dy}, \quad \text{or} \quad \frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}.$$

Hence, equation (2) may be written

$$d^2u = \frac{d^2u}{dx^2}dx^2 + 2\,\frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy^2}dy^2.$$

In the same manner the higher differentials may be obtained.

NOTE.—In the successive differentiation of functions of several independent variables, the following notation is used:

$$\frac{d^{n+m+p+\text{ etc.}}u}{dx^n\,dy^m\,dz^p\text{ etc.}};$$

indicating that $n+m+p+$ etc. differentiations have been performed, n with respect to x, m with respect to y, etc., and it is evident from the preceding demonstration that it is immaterial in what order the operation is performed, provided the entire series of differentiations is effected.

73. Differentiating equation (2) we shall find

$$d^3u = \frac{d^3u}{dx^3}\,dx^3 + 3\,\frac{d^3u}{dx^2\,dy}\,dx^2\,dy + 3\,\frac{d^3u}{dy^2\,dx}\,dy^2\,dx + \frac{d^3u}{dy^3}\,dy^3.$$

$$d^4u = \frac{d^4u}{dx^4}\,dx^4 + 4\,\frac{d^4u}{dx^3\,dy}\,dx^3\,dy + 6\,\frac{d^4u}{dx^2dy^2}\,dx^2\,dy^2$$
$$+ 4\,\frac{d^4u}{dy^3\,dx}\,dy^3\,dx + \frac{d^4u}{dy^4}\,dy^4.$$

. . . .

It will be observed that the co-efficients and indices in the different terms of these differentials are the same as those in the corresponding powers of a binomial, and this law will hold good in all cases.

Hence we may write the following *symbolic formula:*

$$d^nu = \left\{\frac{du}{dx}\,dx + \frac{du}{dy}\,dy\right\}^n,$$

in the development of which by the binomial theorem the exponents are to be placed over the d in du, and over the x and y in dx and dy.

This formula has no siguificance when taken in its literal sense, but it is extremely valuable as an aid to the memory.

74. Problem.—*To differentiate* u = F(v,z), *in which* v *and* z *are functions of both* x *and* y.

Since u is a function of x and y, we have

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy.$$

But since v and z are functions of x and y, we have

$$\frac{du}{dx}dx = \frac{du}{dv}\frac{dv}{dx}dx + \frac{du}{dz}\frac{dz}{dx}dx;$$

$$\frac{du}{dy}dy = \frac{du}{dv}\frac{dv}{dy}dy + \frac{du}{dz}\frac{dz}{dy}dy;$$

$$dv = \frac{dv}{dx}dx + \frac{dv}{dy}dy; \quad dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy.$$

Hence, by substitution and reduction,

$$du = \frac{du}{dv}dv + \frac{du}{dz}dz.$$

A second differentiation will give

$$d^2u = \frac{d^2u}{dv^2}dv^2 + 2\frac{d^2u}{dv\,dz}dv\,dz + \frac{d^2u}{dz^2}dz^2 + \frac{du}{dv}d^2v + \frac{du}{dz}d^2z.$$

The general solution of this problem will be as follows:

Differentiate as though v *and* z *were the only variables, and substitute for* dv, dz, d^2v, d^2z, etc., *their values taken from the equations which connect* v *and* z *with* x *and* y.

75. Implicit functions.—Before showing how the differentiation of implicit functions of two or more variables is effected, we wish to establish the formula for the second derivative of an implicit function of a single variable.

Let $u = F(x,y) = 0$ be such a function. Then, as we have already found [Art. 34],

$$\frac{du}{dy}\frac{dy}{dx} + \frac{du}{dx} = 0 \quad (1).$$

Designating this expression by v, and differentiating, we have

$$\frac{dv}{dy}\frac{dy}{dx} + \frac{dv}{dx} = 0 \quad (2).$$

But $\frac{dv}{dy} = \frac{d^2u}{dy^2}\frac{dy}{dx} + \frac{d^2u}{dx\,dy}$; $\frac{dv}{dx} = \frac{d^2u}{dx\,dy}\frac{dy}{dx} + \frac{du}{dy}\frac{d^2y}{dx^2} + \frac{d^2u}{dx^2}$.

$\therefore$ by substitution in (2),

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy}\frac{dy}{dx} + \frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{du}{dy}\frac{d^2y}{dx^2} = 0 \quad (3).$$

Placing in (3) the value of $\frac{dy}{dx}$ derived from (1), we have, after transposition and reduction,

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2 - 2\frac{d^2u}{dx\,dy}\frac{du}{dx}\frac{du}{dy} + \frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3}.$$

This equation gives the value of $\frac{d^2y}{dx^2}$ in terms of known quantities, but it is so complicated that in practice it is generally better to follow the *method* than to apply the formula itself. The third and higher derivatives may be obtained in a similar manner, but their forms are quite unmanageable.

75′. Problem.—*To differentiate* $\mathrm{u} = \mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z}) = 0$, *in which* z *is an* implicit *function of* x *and* y.

For the first derivative we shall have at once

$$\frac{du}{dz}\frac{dz}{dx}+\frac{du}{dx}=0 \quad (1);$$

$$\frac{du}{dz}\frac{dz}{dy}+\frac{du}{dy}=0 \quad (2);$$

$$\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz=0 \quad (3)$$

Equations (1) and (2) will give the *partial derivatives* of z with respect to x and y, while (3) will give the value of the *total differential* of z.

For the second derivatives we shall find from (1) and (2), as in the last article,

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dz}\frac{dz}{dx}+\frac{d^2u}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{du}{dz}\frac{d^2z}{dx^2}=0 \quad (4).$$

$$\frac{d^2u}{dy^2}+2\frac{d^2u}{dy\,dz}\frac{dz}{dy}+\frac{d^2u}{dz^2}\left(\frac{dz}{dy}\right)^2+\frac{du}{dz}\frac{d^2z}{dy^2}=0 \quad (5).$$

Differentiating (3),

$$\frac{d^2u}{dx^2}dx^2+\frac{d^2u}{dy^2}dy^2+\frac{d^2u}{dz^2}dz^2+2\frac{d^2u}{dx\,dy}dx\,dy+2\frac{d^2u}{dx\,dz}dx\,dz$$
$$+2\frac{d^2u}{dy\,dz}dy\,dz+\frac{du}{dz}d^2z=0 \quad (6);$$

and, differentiating (1) with respect to (y),

$$\frac{d^2u}{dx\,dy}+\frac{d^2u}{dx\,dz}\frac{dz}{dy}+\frac{d^2u}{dz\,dy}\frac{dz}{dx}+\frac{d^2u}{dz^2}\frac{dz}{dy}\frac{dz}{dx}+\frac{du}{dz}\frac{d^2z}{dx\,dy}=0 \quad (7).$$

From equations (4), (5), and (7) we can obtain the values of $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$, and $\frac{d^2z}{dx\,dy}$, in terms of known quantities, while from (6) we find the value of d^2z.

The expressions for the higher derivatives and differentials are too complicated to be presented here; and, fortunately, they are seldom required in practice.

Corollary.—Let $u = F(x, y, z)$ be a *homogeneous function of the n^{th} degree;* that is, such a function that if we multiply each variable by t, the result will be the same as if we had multiplied the function by t^n. Then

$$F(tx, ty, tz) = t^n F(x, y, z).$$

Differentiating with respect to t, we have

$$\frac{dF}{dtx}\frac{dtx}{dt} + \frac{dF}{dty}\frac{dty}{dt} + \frac{dF}{dtz}\frac{dtz}{dt} = nt^{n-1} F(x, y, z).$$

Observing that

$$\frac{dtx}{dt} = x, \quad \frac{dty}{dt} = y, \quad \frac{dtz}{dt} = z,$$

and making $t = 1$, we have

$$x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} = nF(x, y, z);$$

whence it follows that *if we multiply the partial derivatives of a homogeneous function of the* nth *degree by their respective variables, the sum of the products so obtained will be* n *times the function.*

Scholium.—We have confined our investigations in this chapter to functions of two variables, but the student may readily see that the method will be the same for any number of variables whatever; and the fact is that by far the most numerous and important class of problems to which the Calculus is applied involve functions of only one or two independent variables.

76. EXAMPLES.

1. $$u = x^3 y^2 + y^3 x^2.$$

$$\frac{du}{dx} = 3x^2 y^2 + 2xy^3; \quad \frac{du}{dy} = 3y^2 x^2 + 2x^3 y;$$

$$du = 3x^2 y^2 dx + 2xy^3 dx + 3y^2 x^2 dy + 2x^3 ydy.$$

$$\frac{d^2u}{dx^2} = 6xy^2 + 2y^3; \quad \frac{d^2u}{dy^2} = 6x^2 y + 2x^3;$$

$$\frac{d^2u}{dx\, dy} = 6x^2 y + 6xy^2 = \frac{d^2u}{dy\, dx}.$$

$$d^2u = 6xy^2 dx^2 + 6x^2 ydxdy + 2y^3 dx^2 + 6xy^2 dxdy + 6x^2 ydxdy$$
$$+ 2x^3 dy^2 + 6xy^2 dxdy + 6x^2 ydy^2$$
$$= (6xy^2 + 2y^3) dx^2 + 12 (x^2 y + xy^2) dx\, dy + (6x^2 y + 2x^3) dy^2.$$

2. $$u = v^2 z; \quad v = \sqrt{x^2 + y^2}; \quad z = x^2 y^2.$$

We have

$$\frac{du}{dv} = 2vz; \quad \frac{du}{dz} = v^2; \quad dv = \frac{xdx + ydy}{\sqrt{x^2 + y^2}};$$

$$dz = 2x^2 ydy + 2y^2 xdx.$$

$$\therefore\ du = \frac{du}{dv} dv + \frac{du}{dz} dz$$

$$= 2vz \left\{ \frac{xdx + ydy}{\sqrt{x^2 + y^2}} \right\} + 2v^2 (x^2 ydy + y^2 xdx) = ?$$

What is the value of d^2u?

3. $$u = \sin v \cos z; \quad v = 2x + 3y, \quad z = 4x + 5y.$$

We have

$$\frac{du}{dv} = \cos v \cos z; \quad \frac{du}{dz} = -\sin v \sin z;$$

$$dv = 2dx + 3dy; \quad dz = 4dx + 5dy.$$

$$\therefore \ du = \cos v \cos z\,(2dx + 3dy) - \sin v \sin z\,(4dx + 5dy)$$

$$= (2 \cos v \cos z - 4 \sin v \sin z)\,dx$$

$$+ (3 \cos v \cos z - 5 \sin v \sin z)\,dy.$$

Find the value of d^2u.

4. $$u = x^4 + 2ax^2 y - ay^3 = 0.$$

In this example y is an implicit function of x.

We have

$$\frac{du}{dy} = 2ax^2 - 3ay^2; \quad \frac{du}{dx} = 4x^3 + 4axy;$$

$$\frac{d^2u}{dy^2} = -6ay; \quad \frac{d^2u}{dx^2} = 12x^2 + 4ay; \quad \frac{d^2u}{dxdy} = 4ax.$$

$$\therefore \ \frac{dy}{dx} = -\frac{4x^3 + 4axy}{2ax^2 - 3ay^2};$$

$$\frac{d^2y}{dx^2} = -\left\{\begin{array}{l} (12x^2 + 4ay)(2ax^2 - 3ay^2)^2 \\ -8ax\,(4x^3 + 4axy)(2ax^2 - 3ay^2) \\ +(4x^3 + 4axy)^2(-6ay) \end{array}\right\} \div (2ax^2 - 3ay^2)^3 = ?$$

5. $$u = x^3 - 3axy + y^3 = 0.$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Ans. $\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2 - ax)^3}$.

6. $$u = ax^3 + x^3y - ay^3 = 0.$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and show that $\frac{dy}{dx} = 1$ when $x = 0$, $y = 0$.

7. $$u = y^3 + 3x^2 + 2xy - z^2 = 0,$$

in which z is an implicit function of x and y.

We have

$$\frac{du}{dx} = 6x + 2y; \quad \frac{du}{dy} = 3y^2 + 2x; \quad \frac{du}{dz} = -2z;$$

$$\frac{d^2u}{dx^2} = 6; \quad \frac{d^2u}{dy^2} = 6y; \quad \frac{d^2u}{dz^2} = -2;$$

$$\frac{d^2u}{dx\,dy} = 2; \quad \frac{d^2u}{dx\,dz} = 0; \quad \frac{d^2u}{dy\,dz} = 0.$$

These values substituted in the formulas for implicit functions give

$$\frac{dz}{dx} = \frac{3x + y}{z}; \quad \frac{dz}{dy} = \frac{3y^2 + 2x}{2z};$$

$$(6x + 2y)dx + (3y^2 + 2x)dy - 2zdz = 0,$$

from which the value of dz may be found.

Also, $$6 - 2\left\{\frac{3x + y}{z}\right\}^2 - 2z\frac{d^2z}{dx^2} = 0;$$

$$6y - 2\left\{\frac{3y^2 + 2x}{2z}\right\}^2 - 2z\frac{d^2z}{dy^2} = 0;$$

$$6ydy^2 + 6dx^2 + 4dxdy - 2dz^2 - 2zd^2z = 0;$$

from which equations $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$, and d^2z, can be found.

8. $$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

Find $$\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dy^2}, \text{ and } d^2z.$$

9. $$u = ax^2 + by^2 + cz^2 + 2exy + 2fxz + 2hyz.$$

We have

$$\frac{du}{dx} = 2(ax + ey + fz);$$

$$\frac{du}{dy} = 2(by + ex + hz);$$

$$\frac{du}{dz} = 2(cz + fx + hy).$$

$\therefore$ by substitution in the formula for homogeneous functions,

$$x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} = 2x(ax + ey + fz) + 2y(by + ex + hz)$$
$$+ 2z(cz + fx + hy)$$
$$= 2(ax^2 + by^2 + cz^2 + 2exy + 2fxz + 2hyz)$$
$$= 2u.$$

CHAPTER X.

DEVELOPMENT OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

77. Let $u = F(x,y)$, and let it be proposed to develop

$$u_1 = F(x + h,\ y + k)$$

into a series.

The required development may evidently be obtained by expanding

$$F(x + ht,\ y + kt)$$

with reference to t as a new variable, and then making t equal to *unity* in the result.

Differentiating this expression with reference to t, and writing dx, dy, for $d\ (x + ht)$, $d\ (y + kt)$, we have

$$\frac{dF}{dt} = \frac{dF}{dx}h + \frac{dF}{dy}k\ :$$

$$\frac{d^2F}{dt^2} = \frac{d^2F}{dx^2}h^2 + 2\frac{d^2F}{dx\,dy}hk + \frac{d^2F}{dy^2}k^2\ :$$

. . . .

$$\frac{d^nF}{dt^n} = \frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots + \frac{d^nF}{dy^n}k^n.$$

If in these expressions we make $t = 0$, they will become the partial derivatives of $F(x,y)$, and by substitution in Maclaurin's formula, we have

$$F(x+ht,\ y+kt)=\left(F(x,y)\right)+\left(\frac{dF}{dx}h+\frac{dF}{dy}k\right)\frac{t}{1}$$

$$+\left(\frac{d^2F}{dx^2}h^2+2\frac{d^2F}{dx\,dy}hk+\frac{d^2F}{dy^2}k^2\right)\frac{t^2}{1.\,2}\cdots$$

$$\cdots+\frac{t^n}{1.\,2\,.\,.\,n}\left\{\frac{d^nF}{dx^n}h^n+\cdot\cdot+\frac{d^nF}{dy^n}k^n\right\},$$

in the last term of which $x+\theta ht$, $y+\theta kt$, must be substituted for x and y, in accordance with the limitations of Maclaurin's formula.

Now, making $t=1$ in this equation, we have

$$F(x+h,\ y+k)$$

$$=\left(F(x,y)\right)+\frac{dF}{dx}h+\frac{dF}{dy}k+\frac{d^2F}{dx^2}\frac{h^2}{1.\,2}+\frac{d^2F}{dx\,dy}hk$$

$$+\frac{d^2F}{dy^2}\frac{k^2}{1.\,2}+\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{d^{n-1}F}{dx^{n-1}}\frac{h^{n-1}}{1.\,2\,.\,.\,(n-1)}+\frac{d^{n-1}F}{dx^{n-2}dy}\frac{h^{n-2}k}{1.\,2\,.\,.\,.\,(n-2)}$$

$$\cdots\cdots+\frac{d^{n-1}F}{dy^{n-1}}\frac{k^{n-1}}{1.\,2\,.\,.\,.\,(n-1)}$$

$$+\left\{\frac{d^nF}{dx^n}h^n+\cdots+\frac{d^nF}{dy^n}k^n\right\}\cdot\frac{1}{1.\,2\,.\,.\,.\,n}\qquad(1),$$

x and y being replaced in the last term by $x+\theta h$, $y+\theta k$.

The application of this formula is restricted to those cases in which the first n derivatives are continuous between x and $x+h$, y and $y+k$.

If the terms contained in the { } tend toward zero as n increases, then formula (1) will give the *exact* development of $F(x+h,\ y+k)$, and may be considered as the

extension of Taylor's formula to functions of two independent variables.

78. If in (1) we make $x = 0$, $y = 0$, and replace h and k by x and y, we shall have

$$
\begin{aligned}
F(x,y) = F(0,0) &+ \frac{dF}{dx}\frac{x}{1} + \frac{d^2F}{dx^2}\frac{x^2}{1.\,2} + \cdots \\
&+ \frac{dF}{dy}\frac{y}{1} + \frac{d^2F}{dy^2}\frac{y^2}{1.\,2} + \cdots \\
&+ \frac{d^2F}{dx\,dy}xy + \cdots \qquad\qquad (2) \\
&\qquad + \left\{\frac{d^nF}{dx^n}\frac{x^n}{1.\,2\,.\,.\,n} + \cdots \right. \\
&\qquad\qquad \left. \frac{d^nF}{dy^n}\frac{y^n}{1.\,2\,.\,.\,n}\right\},
\end{aligned}
$$

x and y in the $\{\ \}$ being replaced by θx, θy.

If the terms in the $\{\ \}$ tend toward zero as n increases, then formula (2) will give the *exact* development of $F(x,y)$, and may be considered as the **extension of Maclaurin's formula** to functions of two independent variables.

It is to be understood that in this formula the values of the various derivatives are to be found under the supposition that $x = 0$ and $y = 0$ after differentiating.

79. EXAMPLE.

Develop $u = e^x \sin y$.

We have

$$\frac{du}{dx} = e^x \sin y = \frac{d^2u}{dx^2} = \frac{d^3u}{dx^3}; \text{ etc.}$$

$$\frac{du}{dy} = e^x \cos y; \qquad \frac{d^2u}{dy^2} = -e^x \sin y;$$

$$\frac{d^3u}{dy^3} = -e^x \cos y; \qquad \frac{d^4u}{dy^4} = e^x \sin y, \text{ etc.}$$

$\therefore$ making $x = 0$ and $y = 0$,

$$(u) = F(0,0) = 0; \quad \frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 1, \text{ etc., etc.}$$

$\therefore$ by substitution in (2),

$$u = e^x \sin y = y + xy + \frac{x^2y}{1.\,2} - \frac{y^3}{1.\,2.\,3}, \text{ etc.}$$

80. Problem. — Given $u = F(y)$, and $y = z + xf(y)$ in which x and z are independent variables; to develop u in terms of x.

Since u is a function of x, we have, by Maclaurin's formula,

$$u = F(y) = (u) + \left(\frac{du}{dx}\right)\frac{x}{1} + \left(\frac{d^2u}{dx^2}\right)\frac{x^2}{1.\,2} + \text{etc.};$$

and it remains to determine $(u), \left(\frac{du}{dx}\right)$, etc.

1*st.* Since $y = z + xf(y)$, we have $y = z$ when $x = 0$; hence, $(u) = \big(F(y)\big) = F(z)$.

2*d.* Designating $f(y)$ by Y, we have $y = z + xY$.

$$\therefore \quad \frac{dy}{dx} = Y + x\frac{dY}{dy}\frac{dy}{dx}; \quad \frac{dy}{dz} = 1 + x\frac{dY}{dy}\frac{dy}{dz}.$$

Eliminating x from these two equations, we have

$$\frac{dy}{dx} = Y\frac{dy}{dz},$$

and consequently,

$$\frac{dF(y)}{dy}\frac{dy}{dx}=\frac{dF(y)}{dx}=Y\frac{dF(y)}{dy}\frac{dy}{dz}=Y\frac{dF(y)}{dz},$$

a relation which is independent of the character of $F(y)$.

If in this equation we make $F(y)=Y^n$, it will become

$$\frac{dY^n}{dx}=Y\frac{dY^n}{dz}.$$

Now we have

$$\frac{d}{dx}\left\{Y^n\frac{dF(y)}{dz}\right\}$$

$$=\frac{dF(y)}{dz}\frac{dY^n}{dx}+Y^n\frac{d^2F(y)}{dz\,dx}=Y\frac{dF(y)}{dz}\frac{dY^n}{dz}+Y^n\frac{d^2F(y)}{dz\,dx}$$

$$=\frac{dF(y)}{dx}\frac{dY^n}{dz}+Y^n\frac{d^2F(y)}{dz\,dx}=\frac{d}{dz}\left\{Y^n\frac{dF(y)}{dx}\right\}$$

$$=\frac{d}{dz}\left\{Y^{n+1}\frac{dF(y)}{dz}\right\}.$$

$\therefore$ making n equal to 1, 2, 3, etc., in succession,

$$\frac{d}{dx}\left\{Y\frac{dF(y)}{dz}\right\}=\frac{d}{dz}\left\{Y^2\frac{dF(y)}{dz}\right\};\quad \frac{d}{dx}\left\{Y^2\frac{dF(y)}{dz}\right\}$$

$$=\frac{d}{dz}\left\{Y^3\frac{dF(y)}{dz}\right\};\text{ etc.}$$

If now we differentiate successively the equation

$$\frac{dF(y)}{dx}=Y\frac{dF(y)}{dz},$$

we shall have

$$\frac{d^2F(y)}{dx^2}=\frac{d}{dx}\left\{Y\frac{dF(y)}{dz}\right\}=\frac{d}{dz}\left\{Y^2\frac{dF(y)}{dz}\right\};$$

$$\frac{d^3F(y)}{dx^3} = \frac{d^2}{dx\,dz}\left\{ Y^2\frac{dF(y)}{dz} \right\} = \frac{d^2}{dz^2}\left\{ Y^3\frac{dF(y)}{dz} \right\};$$

. . . .

$$\frac{d^nF(y)}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left\{ Y^n\frac{dF(y)}{dz} \right\}.$$

Finally, making $x = 0$, whence $(y) = z$, and $f(y)$ or $Y = f(z)$, and substituting the above expressions in the development of u or $F(y)$ by Maclaurin's formula, we have

$$u = F(y) = F(z) + x\left\{ f(z)\ \frac{dF(z)}{dz} \right\}$$
$$+ \frac{x^2}{1.\,2}\frac{d}{dz}\left\{ \big(f(z)\big)^2\frac{dF(z)}{dz} \right\}$$
$$+ \frac{x^3}{1.\,2.\,3}\frac{d^2}{dz^2}\left\{ \big(f(z)\big)^3\frac{dF(z)}{dz} \right\} + \text{etc.},$$

which formula is known as **Lagrange's Theorem.**

Corollary.— If $F(y) = y$, then $F(z) = z$, and $\frac{dF(z)}{dz} = 1$; whence

$$y = z + x\big(f(z)\big) + \frac{x^2}{1.\,2}\frac{d\big(f(z)\big)^2}{dz} + \frac{x^3}{1.\,2.\,3}\frac{d^2\big(f(z)\big)^3}{dz^2} + \text{etc.},$$

a formula for the development of $y = z + xf(y)$.

These two formulas are of frequent application in the sciences of Mechanics and Physical Astronomy.

81. EXAMPLES.

1. Given $y^3 - ay + b = 0$, or $y = \frac{b}{a} + \frac{1}{a}y^3$:

to expand y in terms of $\frac{1}{a}$.

We have

$$x = \frac{1}{a}; \quad z = \frac{b}{a}; \quad f(y) = y^3; \quad f(z) = z^3;$$

$$\frac{d\big(f(z)\big)^2}{dz} = \frac{d(z^6)}{dz} = 6z^5 = 6\left(\frac{b}{a}\right)^5; \quad \frac{d^2\big(f(z)\big)^3}{dz^2} = 8.\,9.\left(\frac{b}{a}\right)^7; \text{ etc.}$$

Hence, by substitution in Lagrange's theorem (Cor.),

$$y = \left(\frac{b}{a}\right) + \left(\frac{1}{a}\right)\left(\frac{b}{a}\right)^3 + 3\left(\frac{1}{a^2}\right)\left(\frac{b}{a}\right)^5 + 12\left(\frac{1}{a^3}\right)\left(\frac{b}{a}\right)^7 + \text{etc.}$$

$$= \frac{b}{a}\left\{1 + \frac{b^2}{a^3} + 3\,\frac{b^4}{a^6} + 12\,\frac{b^6}{a^9} + \text{etc.}\right\}.$$

We may in a similar manner expand the unknown quantity in any algebraic equation, and thus approximate to the roots of the equation.

2. $y = b + ca^y$. Expand y in terms of c.

Here $z = b$; $x = c$; $f(y) = a^y$; $f(z) = a^b$.

$$\therefore \frac{d\big(f(z)\big)^2}{dz} = 2 \log a.\ a^{2b}; \quad \frac{d^2\big(f(z)\big)^3}{dz^2} = 3^2 \log^2 a.\ a^{3b}; \text{ etc.}$$

$$\therefore y = b + ca^y = b + ca^b + 2 \log a.\ \frac{c^2}{1.\,2}\,a^{2b}$$

$$+ 3^2 \log^2 a.\ \frac{c^3}{1.\,2.\,3}\,a^{3b}; \text{ etc.}$$

Corollary 1.—If $a = e$ and $b = 1$, then

$$y = 1 + ce^y = 1 + ce + 2e^2\,\frac{c^2}{1.\,2} + 3^2\,\frac{e^3\,c^3}{1.\,2.\,3} + \text{etc.}$$

Corollary 2.—If $e=1$, whence $y=1+e^y$, or $y=\log(y-1)$, then

$$y=\log(y-1)=1+e+2.\frac{e^2}{1.\,2}+3^2.\frac{e^3}{1.\,2.\,3}+\text{etc.}$$

3. $y=a+x\log y$. Expand y in terms of x.

We have

$$z=a;\quad x=x;\quad f(y)=\log(y);\quad f(z)=\log(a);$$

$$\therefore\ \frac{d(f(z))^2}{dz}=\frac{2\log(a)}{a};\ \frac{d^2(f(z))^3}{dz^2}=\frac{3\log a}{a^2}(2-\log a);\ \text{etc.}$$

$$\therefore\ y=a+x\log y=a+x\log a+\frac{2\log a}{a}\frac{x^2}{1.\,2}$$

$$+\frac{3\log a}{a^2}(2-\log a)\frac{x^3}{1.\,2.\,3}+\text{etc.}$$

4. Develop $y=z+e\sin y$.

We have

$$x=e;\quad f(y)=\sin(y);\quad f(z)=\sin z;\quad (f(z))^2=\sin^2 z;\ \text{etc.}$$

$$\therefore\ \frac{d(f(z))^2}{dz}=2\sin z\cos z=\sin 2z;$$

$$\frac{d^2(f(z))^3}{dz^2}=6\sin z\cos^2 z-3\sin^3 z$$

$$=3\sin z\{2\cos^2 z-\sin^2 z\}=3\sin z\{\cos^2 z+\cos 2z\}$$

$$=\frac{9}{4}\sin 3z-\frac{3}{4}\sin z;\ \text{etc.}$$

$$\therefore\ y=z+e\sin y=z+e\sin z+\frac{e^2}{1.\,2}\sin 2z$$

$$+\frac{e^3}{1.\,2.\,3}\left\{\frac{9}{4}\sin 3z-\frac{3}{4}\sin z\right\}+\text{etc.}$$

5. Given $u = \sin y$; $y = z + e \sin y$. Develop u.

We have

$$F(y) = \sin y; \quad F(z) = \sin z; \quad f(z) = \sin z; \quad x = e.$$

$$\therefore \; f(z)\,\frac{dF(z)}{dz} = \sin z \cos z = \frac{1}{2}\sin 2z;$$

$$\frac{d}{dz}\left\{ (f(z))^2 \frac{dF(z)}{dz} \right\} = \frac{d}{dz}(\sin^2 z \cdot \cos z) = \frac{3}{4}\sin 3z - \frac{1}{4}\sin z;$$

. . . .

$\therefore$ by substitution in Lagrange's theorem,

$$u = \sin y = \sin z + \frac{1}{2} e \sin 2z + \frac{e^2}{1.\,2}\left\{ \frac{3}{4}\sin 3z - \frac{1}{4}\sin z \right\} + \text{etc.}$$

CHAPTER XI.

MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

82. If $F(x, y)$ be a function of two independent variables, such that for the particular values $x_0\ y_0$ of x and y, $F(x_0, y_0)$ is greater than $F(x_0 + h,\ y_0 + k)$, then $F(x_0, y_0)$ is said to be a **maximum** of $F(x, y)$; and if $F(x_0, y_0)$ be less than $F(x_0 + h,\ y_0 + k)$, $F(x_0, y_0)$ is said to be a **minimum** of $F(x, y)$.

In order to determine the tests for these two cases let us designate

$$h \text{ by } adx; \quad k \text{ by } ady; \quad F(x+h,\, y+k) \text{ by } f(a);$$
$$\text{and } F(x,y) \text{ by } f(0).$$

Then $F(x + adx,\, y + ady) - F(x,y) = f(a) - f(0)$.

Now it is evident that in order that $F(x,y)$ may be a maximum, we must have

$$F(x + adx,\, y + ady) - F(x,y) < 0;$$

and that $F(x,y)$ may be a minimum,

$$F(x + adx,\, y + ady) - F(x,y) > 0.$$

That is, for a *maximum*, $f(a) - f(0) < 0$; and for a *minimum*, $f(a) - f(0) > 0$.

These conditions require that for a maximum of $F(x,y)$, $f(0)$ must be a maximum of $f(a)$; and for a minimum of $F(x,y)$, $f(0)$ must be a minimum of $f(a)$.

Now, in order that $f(0)$ may be either a maximum or minimum, we must have, as in the case of functions of one variable,

$$f'(0) = 0 \text{ or } \infty;$$

and, supposing $f(a)$ and its derivatives to be continuous, the first of its derivatives which does not reduce to zero must be of an *even* order, *negative* for a maximum, and *positive* for a minimum.

If we have $f(a) = F(x + adx,\, y + ady)$, then, by differentiating with respect to a, we have

$$f'(a) = \frac{dF}{d(x + adx)}\, dx + \frac{dF}{d(y + ady)}\, dy$$

Making $a = 0$, this reduces to

$$f'(0) = \frac{dF}{dx}\,dx + \frac{dF}{dy}\,dy$$

$$= \frac{du}{dx}\,dx + \frac{du}{dy}\,dy,$$

which is the value of du.

Observing, then, that if $u = F(x, y)$ we shall have

$$f(0) = u; \quad f'(0) = du, \text{ etc.},$$

we obtain the following

Rule.—Form the successive differentials of u. Then,

1*st.* For either a maximum or minimum,

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = 0; \quad \text{whence } \frac{du}{dx} = 0, \ \frac{du}{dy} = 0;$$

$$\text{or } \frac{du}{dx}dx + \frac{du}{dy}dy = \infty.$$

2*d.* $$d^n u = \frac{d^n u}{dx^n}\,dx^n + n\,\frac{d^n u}{dx^{n-1}\,dy}\,dx^{n-1}dy + \cdots \frac{d^n u}{dy^n}\,dy^n$$

must be of an *even* degree and *negative* for a *maximum*, or *positive* for a *minimum.*

On account of the complicated forms of the higher differentials the application of this method is almost entirely restricted to the first and second differentials. Let us suppose, then, that the second differential *does not reduce* to zero for those values of x and y which render du either zero or infinity.

Then we shall have, for a *maximum*,

$$d^2u = \frac{d^2u}{dx^2}\,dx^2 + 2\,\frac{d^2u}{dx\,dy}\,dx\,dy + \frac{d^2u}{dy^2}\,dy^2 < 0;$$

and for a *minimum*,

$$d^2u = \frac{d^2u}{dx^2}\,dx^2 + 2\,\frac{d^2u}{dx\,dy}\,dx\,dy + \frac{d^2u}{dy^2}\,dy^2 > 0.$$

Now, in order that this expression may have constantly the same sign, whatever may be the value of its middle term, we must have, in accordance with the theory of quadratics in Algebra,

$$\left(\frac{d^2u}{dxdy}\right)^2 < \frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2},$$

which requires that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ shall have the same sign; and d^2u will be *negative* if $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ are both *negative*, or *positive* if they are both *positive*.

We have, therefore (to recapitulate), the following tests for determining a maximum or minimum value of $u = F(x, y)$.

1*st*. **For a maximum,**

$$\frac{du}{dx} = 0,\ \frac{du}{dy} = 0;$$

$$\text{or } du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = \infty:$$

$$\frac{d^2u}{dx^2} < 0,\quad \frac{d^2u}{dy^2} < 0;\quad \left(\frac{d^2u}{dxdy}\right)^2 < \frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2}.$$

2*d*. **For a minimum,**

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0;$$

$$\text{or } du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = \infty:$$

$$\frac{d^2u}{dx^2} > 0, \quad \frac{d^2u}{dy^2} > 0; \quad \left(\frac{d^2u}{dx\,dy}\right)^2 < \frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2}.$$

We may remark that the sign of d^2u will depend on the signs of $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dy^2}$, not only when $\left(\frac{d^2u}{dx\,dy}\right)^2 < \frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2}$, but also when the relation between the two terms of this expression is that of equality.

The consideration of this case would involve an additional test for the complete determination of maxima and minima, but the resulting expression is rather complicated, and the case seldom occurs in practice.

83. EXAMPLES.

1. Find the values of x and y which render

$u = x^3 + y^3 - 3axy$ a maximum or minimum.

We have

$$\frac{du}{dx} = 3x^2 - 3ay = 0, \quad \therefore\ y = \frac{x^2}{a}.$$

$$\frac{du}{dy} = 3y^2 - 3ax = 0, \quad \therefore\ x = \frac{y^2}{a}.$$

$$\therefore\ x = 0 \text{ or } a, \text{ and } y = 0 \text{ or } a.$$

$$\text{Also, } \frac{d^2u}{dx^2} = 6x = +; \quad \frac{d^2u}{dy^2} = 6y = +; \quad \frac{d^2u}{dx\,dy} = -3a;$$

$$\left(\frac{d^2u}{dx\,dy}\right)^2 - \frac{d^2u}{dx^2}\frac{d^2u}{dy^2} = + \quad 9a^2 \text{ for } x = 0,\ y = 0,$$

$$= -27a^2 \text{ for } x = a,\ y = a.$$

$\therefore\ x = a,\ y = a$, render the value of u a minimum.

2. $$u = x^2 + xy + y^2 + \frac{a^3}{x} + \frac{a^3}{y}.$$

We have

$$\frac{du}{dx} = 2x + y - \frac{a^3}{x^2} = 0;\quad \frac{du}{dy} = 2y + x - \frac{a^3}{y^2} = 0.$$

From these two equations we derive $x = y$, and therefore

$$x = y = \frac{a}{\sqrt[3]{3}}.$$

Also, $$\frac{d^2u}{dx^2} = 2 + \frac{2a^3}{x^3};\quad \frac{d^2u}{dx\,dy} = 1;\quad \frac{d^2u}{dy^2} = 2 + \frac{2a^3}{y^3}.$$

$\therefore$ when $x = y = \dfrac{a}{\sqrt[3]{3}}$, we have

$$\frac{d^2u}{dx^2} = +,\ \frac{d^2u}{dy^2} = +;\ \left(\frac{d^2u}{dx\,dy}\right)^2 < \frac{d^2u}{dx^2}\frac{d^2u}{dy^2};$$

hence the above values of x and y render u a minimum.

3. Find the values of x and y which will render $u = x^4 + y^4 - 4axy^2$ a maximum or minimum.

4. Given the surface of a rectangular parallelopipedon; find its edges when the content is a maximum.

Let x, y, z be the edges, and denote the surface by $6a^2$.

Then $2(xy + xz + yz) = 6a^2$, and $u = xyz$, a maximum.

We have

$$z = \frac{3a^2 - xy}{x + y};$$

$$\therefore\ u = \frac{xy(3a^2 - xy)}{x + y};$$

$$\frac{du}{dx} = \frac{d}{dx}\left\{\frac{3a^2xy - x^2y^2}{x + y}\right\} = 0; \text{ whence } x^2 + 2xy = 3a^2 \ . \ . \ (a)$$

$$\frac{du}{dy} = \frac{d}{dy}\left\{\frac{3a^2xy - x^2y^2}{x + y}\right\} = 0; \text{ whence } y^2 + 2xy = 3a^2 \ . \ . \ (b).$$

From equations (a) and (b) we have

$$x = y = a, \text{ and, therefore, } z = x = y,$$

and the figure is a cube.

NOTE.—In examples where there is no doubt about the existence of a maximum or minimum, it is not necessary to form the higher derivatives.

5. Given the content a^3 of a rectangular parallelopipedon; find its figure when the surface is a minimum.

We have, as in the preceding example,

$$xyz = a^3, \text{ and } u = 2(xy + xz + yz), \text{ a minimum.}$$

It is easily shown that the figure is a cube.

6. Find the edges of the maximum parallelopipedon which can be inscribed in a given ellipsoid.

Let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ . \ . \ . \qquad (1)$$

be the equation of the ellipsoid, referred to its center and axes.

Let x, y, z, be the coördinates of an angle of the parallelopipedon. Then $2x$, $2y$, $2z$, will be its edges, and we shall have

$$u = 8xyz, \text{ a maximum.}$$

We have

$$\frac{du}{dx} = 8yz + 8xy\frac{dz}{dx} = 0, \qquad (2)$$

$$\frac{du}{dy} = 8xz + 8xy\frac{dz}{dy} = 0, \qquad (3)$$

But, by differentiating (1),

$$\frac{dz}{dx} = -\frac{x}{z}\frac{c^2}{a^2}; \qquad \frac{dz}{dy} = -\frac{y}{z}\frac{c^2}{b^2}.$$

$\therefore$ by substitution in (2) and (3), and reducing,

$$z - \frac{x^2}{z}\frac{c^2}{a^2} = 0, \text{ and } z - \frac{y^2}{z}\frac{c^2}{b^2} = 0.$$

Whence, $$\frac{z^2}{c^2} = \frac{x^2}{a^2} = \frac{y^2}{b^2}.$$

$$\therefore \frac{3x^2}{a^2} = 1, \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.$$

7. Find the maximum value of

$$u = ax + by + cz, \text{ in which } x^2 + y^2 + z^2 = 1.$$

Proceeding as in the last example, we find

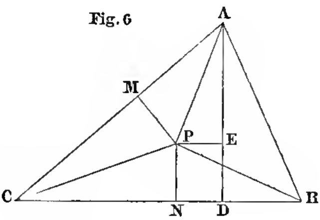
$$x = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad y = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$z = \frac{c}{\sqrt{a^2 + b^2 + c^2}}, \qquad u = \sqrt{a^2 + b^2 + c^2}.$$

8. Find the point in the surface of a triangle from which, if lines be drawn to the angular points, the sum of their squares shall be a minimum.

Let ABC be the triangle, and let P be the required point. Designate the sides by a, b, c. Draw PN and AD perpendicular to BC, and draw AP, CP, BP.

Fig. 6



$$\text{Let } CN = x, \ PN = y.$$

Then $CP^2 = y^2 + x^2 : BP^2 = y^2 + (a - x)^2 :$

$$AP^2 = PE^2 + AE^2 = (CD - CN)^2 + (AD - PN)^2$$

$$= (b \cos C - x)^2 + (b \sin C - y)^2.$$

$\therefore \ u = CP^2 + BP^2 + AP^2$

$$= 3x^2 + 3y^2 + a^2 - 2ax + b^2 - 2bx \cos C - 2by \sin C.$$

We find, by differentiation,

$$x = \tfrac{1}{3}(a + b \cos C) : y = \tfrac{1}{3} b \sin C.$$

which values fix the position of the point P. This point is the center of gravity of the triangle.

9. Given $\quad u = \dfrac{a}{\sin \theta} + \dfrac{b}{\sin \phi} + \dfrac{c}{\sin \psi}$; and

$$a \cot \theta + b \cot \phi + c \cot \psi = \text{constant};$$

to find the values of θ, ϕ, ψ, which shall render u a minimum.

Considering ψ as a function of θ and ϕ, we easily find that

$$\theta = \phi = \psi.$$

CHAPTER XII.

CHANGE OF THE INDEPENDENT VARIABLE, AND ELIMINATION.

84. As in Analytical Geometry it is frequently convenient to effect a transformation of coördinates, so in the Calculus we may often facilitate our operations by changing the independent variable, the necessary formulas for which we will now establish.

85. Problem.—Given $y = F(x)$; to find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, in terms of the derivatives of y and x, with respect to a new variable, t.

We have, at once,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \text{ or } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad . \quad . \quad . \qquad (1).$$

Differentiating again,

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{d^2x}{dt^2}\frac{dy}{dx}; \text{ whence}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dx}}{\left(\frac{dx}{dt}\right)^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3} \quad . \quad . \quad . \qquad (2).$$

In a similar manner the higher derivatives may be found.

Corollary.—If $t = y$, then

$$\frac{dy}{dt} = 1; \quad \frac{d^2y}{dt^2} = 0.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}; \quad \frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3},$$

formulas for changing the independent variable from x to y.

Examples.

1. Given $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + n^2y = 0$, and $x = \cos t$;

transform this equation to one in which t is the independent variable.

We have

$$1 - x^2 = 1 - \cos^2 t = \sin^2 t; \quad \frac{dx}{dt} = -\sin t; \quad \frac{d^2x}{dt^2} = -\cos t.$$

$\therefore$ by substitution in formulas (1) and (2),

$$\frac{dy}{dx} = -\frac{\dfrac{dy}{dt}}{\sin t}; \quad \frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dt^2}\sin t - \dfrac{dy}{dt}\cos t}{\sin^3 t}.$$

Substituting these values in the given equation it becomes

$$\frac{d^2y}{dt^2} + n^2 y = 0.$$

2. $$u + \frac{1}{x}\frac{du}{dx} + \frac{d^2u}{dx^2} = 0;$$

transform to the independent variable t, where $x^2 = 4t$.

We have

$$\frac{du}{dx} = \frac{du}{dt}\,\frac{dt}{dx} = \frac{x}{2}\,\frac{du}{dt};$$

$$\frac{d^2u}{dx^2} = \frac{1}{2}\frac{du}{dt} + \frac{x}{2}\frac{d^2u}{dt^2}\frac{dt}{dx} = \frac{1}{2}\frac{du}{dt} + \frac{x^2}{4}\frac{d^2u}{dt^2}$$

$$= \frac{1}{2}\frac{du}{dt} + t\frac{d^2u}{dt^2}.$$

These values substituted in the given equation reduce it to

$$u + \frac{du}{dt} + t\frac{d^2u}{dt^2} = 0.$$

3. Transform the formula,

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx} + (y - a)\frac{d^2y}{dx^2} = 0,$$

into an equivalent expression in which y is the independent variable.

4. Given $x^2\dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + u = 0$, and $\log x = y$;

transform to the independent variable y.

We have from formula (1),

$$\frac{du}{dx} = \frac{1}{x}\frac{du}{dy}; \quad \therefore \; x\frac{du}{dx} = \frac{du}{dy};$$

and from (2),

$$\frac{d^2u}{dx^2} = \frac{x\dfrac{d^2u}{dy^2} - x\dfrac{du}{dy}}{x^3};$$

whence

$$x^2\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} - \frac{du}{dy}.$$

$\therefore$ by substitution in the given equation, $\dfrac{d^2u}{dy^2} + u = 0.$

5. Given $\dfrac{d^2y}{dx^2}+\dfrac{2x}{1+x^2}\dfrac{dy}{dx}+\dfrac{y}{(1+x^2)^2}=0$, and $x=\text{tang }\theta$;

transform to the independent variable θ.

Ans. $\dfrac{d^2y}{d\theta^2}+y=0.$

6. Given $u=\dfrac{x\dfrac{dy}{dx}-y}{x+y\dfrac{dy}{dx}}$;

transform this into an equivalent expression in terms of r and θ, having given

$$x=r\cos\theta,\ y=r\sin\theta.$$

Taking r as the new independent variable, and θ as a function of r, we must express $\dfrac{dy}{dx}$ in terms of $\dfrac{d\theta}{dr}$.

Now, we have

$$\frac{dy}{dx}=\frac{\dfrac{dy}{dr}}{\dfrac{dx}{dr}};\ \frac{dy}{dr}=\sin\theta+r\cos\theta\frac{d\theta}{dr};$$

$$\frac{dx}{dr}=\cos\theta-r\sin\theta\frac{d\theta}{dr}.$$

.·. by substitution in the given expression

$$u=\frac{r\cos\theta\left\{\dfrac{\sin\theta+r\cos\theta\dfrac{d\theta}{dr}}{\cos\theta-r\sin\theta\dfrac{d\theta}{dr}}\right\}-r\sin\theta}{r\cos\theta+r\sin\theta\left\{\dfrac{\sin\theta+r\cos\theta\dfrac{d\theta}{dr}}{\cos\theta-r\sin\theta\dfrac{d\theta}{dr}}\right\}}$$

$$= \frac{r \cos\theta \sin\theta + r^2 \cos^2\theta \frac{d\theta}{dr} - r\cos\theta\sin\theta + r^2\sin^2\theta\frac{d\theta}{dr}}{r\cos^2\theta - r^2\sin\theta\cos\theta\frac{d\theta}{dr} + r\sin^2\theta + r^2\sin\theta\cos\theta\frac{d\theta}{dr}}$$

$$= r\frac{d\theta}{dr}.$$

This is an important transformation in the theory of curves.

7. Given $u = \dfrac{x\dfrac{dy}{dx} - y}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}$;

transform this into an equivalent expression in terms of r and θ, having given

$$x = r\cos\theta,\ y = r\sin\theta.$$

The values of $\frac{dy}{dx}$, $\frac{dx}{dr}$, $\frac{dy}{dr}$, are the same as in the last example. We therefore have, by substitution in the given expression

$$u = \frac{r\cos\theta\sin\theta + r^2\cos^2\theta\frac{d\theta}{dr} - r\cos\theta\sin\theta + r^2\sin^2\theta\frac{d\theta}{dr}}{\sqrt{\left(\cos\theta - r\sin\theta\frac{d\theta}{dr}\right)^2 + \left(\sin\theta + r\cos\theta\frac{d\theta}{dr}\right)^2}}$$

$$= \frac{r^2\dfrac{d\theta}{dr}}{\sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \frac{r^2}{\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}};$$

another important transformation.

8. Given $(a+y)^3\dfrac{d^3u}{dy^3} + 3(a+y)^2\dfrac{d^2u}{dy^2} + (a+y)\dfrac{du}{dy} + bu = 0$;

transform to the independent variable x, where

$$x = \log (a + y).$$

In making this transformation we shall find it best to apply the *method* rather than the formulas, simplifying as we proceed.

Thus, we have

$$dx = \frac{dy}{a+y}, \quad \text{or} \quad (a+y)\frac{dx}{dy} = 1.$$

$$\therefore \quad (a+y)\frac{du}{dy} = (a+y)\frac{du}{dx}\frac{dx}{dy} = \frac{du}{dx}.$$

Differentiating again, and multiplying by $(a + y)$, we have

$$(a+y)^2\frac{d^2u}{dy^2} + (a+y)\frac{du}{dy} = \frac{d^2u}{dx^2}.$$

Differentiating again, and multiplying by $(a + y)$, we find

$$(a+y)^3\frac{d^3u}{dy^3} + 3(a+y)^2\frac{d^2u}{dy^2} + (a+y)\frac{du}{dy} = \frac{d^3u}{dx^3}.$$

$\therefore$ by substitution in the given expression $\dfrac{d^3u}{dx^3} + bu = 0.$

86. Problem.—Given $u = F(x, y)$, $x = f(r, \theta)$, $y = \phi(r, \theta)$; to find $\dfrac{du}{dx}$, $\dfrac{du}{dy}$, in terms of r and θ.

We have

$$\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta},$$

$$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr}.$$

We find, from these equations,

$$\frac{du}{dx} = \frac{\frac{du}{dr}\frac{dy}{d\theta} - \frac{du}{d\theta}\frac{dy}{dr}}{\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dx}{d\theta}\frac{dy}{dr}} \quad . \quad . \quad . \quad (1),$$

$$\frac{du}{dy} = \frac{\frac{du}{dr}\frac{dx}{d\theta} - \frac{du}{d\theta}\frac{dx}{dr}}{\frac{dy}{dr}\frac{dx}{d\theta} - \frac{dy}{d\theta}\frac{dx}{dr}} \quad . \quad . \quad . \quad (2).$$

As an example of the application of these formulas, let

$x = r \cos\theta$, $y = r \sin\theta$. Then

$$\frac{dx}{dr} = \cos\theta; \ \frac{dy}{dr} = \sin\theta; \ \frac{dx}{d\theta} = -r\sin\theta; \ \frac{dy}{d\theta} = r\cos\theta.$$

$$\therefore \ \frac{du}{dx} = \frac{r\cos\theta\frac{du}{dr} - \sin\theta\frac{du}{d\theta}}{r(\cos^2\theta + \sin^2\theta)} = \cos\theta\frac{du}{dr} - \frac{\sin\theta}{r}\frac{du}{d\theta};$$

$$\frac{du}{dy} = \sin\theta\frac{du}{dr} + \frac{\cos\theta}{r}\frac{du}{d\theta}.$$

87. Problem.—Given $u = F(x, y, z)$, in which x, y, z, are functions of r, θ, ϕ; to find $\frac{du}{dx}$, etc., in terms of r, θ, ϕ.

The following formulas will resolve the problem:

$$\left.\begin{aligned}
\frac{du}{dx} &= \frac{du}{dr}\frac{dr}{dx} + \frac{du}{d\theta}\frac{d\theta}{dx} + \frac{du}{d\phi}\frac{d\phi}{dx}, \\
\frac{du}{dy} &= \frac{du}{dr}\frac{dr}{dy} + \frac{du}{d\theta}\frac{d\theta}{dy} + \frac{du}{d\phi}\frac{d\phi}{dy}, \\
\frac{du}{dz} &= \frac{du}{dr}\frac{dr}{dz} + \frac{du}{d\theta}\frac{d\theta}{dz} + \frac{du}{d\phi}\frac{d\phi}{dz},
\end{aligned}\right\} \quad (1).$$

We may also make use of the following formulas to resolve this problem:

$$\left.\begin{aligned}\frac{du}{dr}&=\frac{du}{dx}\,\frac{dx}{dr}+\frac{du}{dy}\,\frac{dy}{dr}+\frac{du}{dz}\,\frac{dz}{dr},\\ \frac{du}{d\theta}&=\frac{du}{dx}\,\frac{dx}{d\theta}+\frac{du}{dy}\,\frac{dy}{d\theta}+\frac{du}{dz}\,\frac{dz}{d\theta},\\ \frac{du}{d\phi}&=\frac{du}{dx}\,\frac{dx}{d\phi}+\frac{du}{dy}\,\frac{dy}{d\phi}+\frac{du}{dz}\,\frac{dz}{d\phi},\end{aligned}\right\} \quad (2).$$

Let us take as an example

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

From these equations we easily obtain

$$r=\sqrt{x^2+y^2+z^2};\quad \text{tang}\,\theta=\frac{\sqrt{x^2+y^2}}{z};\quad \text{tang}\,\phi=\frac{y}{x}.$$

$$\therefore\quad \frac{dr}{dx}=\frac{x}{r}=\sin\theta\cos\phi;\quad \frac{dr}{dy}=\frac{y}{r}=\sin\theta\sin\phi;$$

$$\frac{dr}{dz}=\frac{z}{r}=\cos\theta;\quad \frac{d\theta}{dx}=\frac{z}{x^2+y^2+z^2}\,\frac{x}{\sqrt{x^2+y^2}}=\frac{\cos\theta\cos\phi}{r};$$

$$\frac{d\theta}{dy}=\frac{\cos\theta\sin\phi}{r};\quad \frac{d\theta}{dz}=-\frac{\sin\theta}{r};\quad \frac{d\phi}{dx}=-\frac{\sin\phi}{r\sin\theta};$$

$$\frac{d\phi}{dy}=\frac{\cos\phi}{r\sin\theta};\quad \frac{d\phi}{dz}=0.$$

$\therefore$ by substitution in (1),

$$\frac{du}{dx}=\frac{du}{dr}\sin\theta\cos\phi+\frac{du}{d\theta}\,\frac{\cos\theta\cos\phi}{r}-\frac{du}{d\phi}\,\frac{\sin\phi}{r\sin\theta},$$

$$\frac{du}{dy} = \frac{du}{dr} \sin \theta \sin \phi + \frac{du}{d\theta} \frac{\cos \theta \sin \phi}{r} + \frac{du}{d\phi} \frac{\cos \phi}{r \sin \theta},$$

$$\frac{du}{dz} = \frac{du}{dr} \cos \theta - \frac{du}{d\theta} \frac{\sin \theta}{r}.$$

We might easily obtain the values of $\frac{du}{dr}$, etc., from (2), by finding from the given relations between x, y, z, r, θ, ϕ, the values of $\frac{dx}{dr}$, etc., and substituting them in (2).

We leave this as an exercise for the student.

88. Elimination. — When in an equation there are two or more variables, and a number of constants, we may, by successive differentiation, obtain a series of new equations by means of which the constants may be eliminated, and thus arrive at an equation containing only variables and their derivatives. Such an equation is called a **differential equation,** and we here present a few examples by way of illustration.

1. Eliminate the constants from the equation

$$y = ax^2 + bx.$$

$$\frac{dy}{dx} = 2ax + b; \quad \frac{d^2y}{dx^2} = 2a.$$

$$\therefore \quad a = \frac{1}{2} \frac{d^2y}{dx^2}; \quad b = \frac{dy}{dx} - x \frac{d^2y}{dx^2};$$

$$y = \frac{1}{2} x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - x^2 \frac{d^2y}{dx^2}, \text{ or}$$

$$2\left(y - x \frac{dy}{dx}\right) + x^2 \frac{d^2y}{dx^2} = 0.$$

2. $$y = m\cos(rx + a).$$

Two differentiations give

$$\frac{d^2y}{dx^2} = -r^2 m\cos(rx + a) = -r^2 y.$$

$$\therefore \quad \frac{d^2y}{dx^2} + r^2 y = 0.$$

3. $$(x-a)^2 + (y-b)^2 = r^2.$$

We have, by differentiation,

$$(x-a) + (y-b)\frac{dy}{dx} = 0,$$

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0,$$

$$\therefore \quad (y-b) = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}; \quad (x-a) = \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\,\frac{dy}{dx}.$$

These values substituted in the given equation reduce it to

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2.$$

The given equation in this example is the well-known equation of the circle, and the result obtained is the differential equation of the same curve.

4. $$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Multiplying by e^x we have

$$y = \frac{e^{2x}+1}{e^{2x}-1}, \text{ whence } e^{2x} = \frac{y+1}{y-1}, \text{ and}$$

$$2x = \log(y+1) - \log(y-1).$$

We obtain, by a single differentiation,

$$\frac{dy}{dx} + y^2 = 1.$$

5. $$z = ax + by + c.$$

Considering y and z as functions of x, and differentiating with respect to x, we find

$$\left(\frac{d^3z}{dx^3}\right)\left(\frac{d^2y}{dx^2}\right) = \left(\frac{d^3y}{dx^3}\right)\left(\frac{d^2z}{dx^2}\right).$$

89. If we have $u = F(z)$, and $z = f(x, y)$, in which x and y are independent, and F is an arbitrary or unknown function, it is possible, by differentiation and elimination, to obtain a new equation in which F does not appear.

Thus we have

$$\frac{du}{dx} = \frac{du}{dz}\,\frac{dz}{dx}; \quad \frac{du}{dy} = \frac{du}{dz}\,\frac{dz}{dy};$$

whence

$$\frac{\frac{du}{dx}}{\frac{du}{dy}} = \frac{\frac{dz}{dx}}{\frac{dz}{dy}}, \text{ or } \frac{du}{dx}\,\frac{dz}{dy} = \frac{du}{dy}\,\frac{dz}{dx},$$

an equation free from the function F.

The following examples will serve to illustrate this operation:

1. Given $u = F(z), \quad z = \dfrac{y^2 - x^2}{x}$.

We have

$$\frac{dz}{dy} = \frac{2y}{x}; \quad \frac{dz}{dx} = -\frac{x^2 + y^2}{x^2}.$$

$\therefore$ by substitution and reduction

$$2xy\frac{du}{dx} + (x^2 + y^2)\frac{du}{dy} = 0.$$

2. $$u = \frac{1}{x}F\left(\frac{x}{y}\right).$$

We have

$$\frac{du}{dx} = -\frac{1}{x^2}F\left(\frac{x}{y}\right) + \frac{1}{xy}F'\left(\frac{x}{y}\right):$$

$$\frac{du}{dy} = -\frac{1}{y^2}F'\left(\frac{x}{y}\right).$$

$$\therefore \quad u + x\frac{du}{dx} + y\frac{du}{dy} = 0.$$

3. $$u = x^n F\left(\frac{y}{x}\right) + y^n f\left(\frac{y}{x}\right).$$

We have

$$\frac{du}{dx} = nx^{n-1}F\left(\frac{y}{x}\right) - yx^{n-2}F'\left(\frac{y}{x}\right) - \frac{y^{n+1}}{x^2}f'\left(\frac{y}{x}\right),$$

$$\frac{du}{dy} = x^{n-1}F'\left(\frac{y}{x}\right) + ny^{n-1}f\left(\frac{y}{x}\right) + \frac{y^n}{x}f'\left(\frac{y}{x}\right).$$

$$\therefore \quad x\frac{du}{dx} + y\frac{du}{dy} - nu = 0.$$

4. $$u = F(x + at) + f(x - at).$$

Differentiating twice, we find

$$\frac{d^2u}{dx^2} = F''(x + at) + f''(x - at),$$

$$\frac{d^2u}{dt^2} = a^2F''(x + at) + a^2f''(x - at).$$

$$\therefore \frac{d^2u}{dt^2} = a^2 \frac{d^2u}{dx^2}.$$

The subjects treated in this chapter are important, but our limits do not admit of any further consideration of them.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO GEOMETRY.

CHAPTER XIII.

TANGENTS AND NORMALS TO PLANE CURVES.

90. The finite equation of a plane curve, being the expression of a relation between the coördinates of its points, may be considered as a relation between a function and its independent variable, and may be written

$$y = F(x); \quad F(x, y) = 0; \text{ etc.}$$

By applying the principles which have been discussed in the preceding chapters, the theory of curves has been generalized and extended far beyond the capability of the ordinary algebraic methods employed in Analytical Geometry. In the remaining chapters of the Differential Calculus we shall present some of the more important applications of the science to the solution of geometrical problems.

91. Problem.—*To find the general differential equation of a tangent line to a plane curve.*

If x', y' be the rectangular coördinates of the point of tangency, the equation of the tangent line will be of the form

$$y - y' = a(x - x'),$$

in which a is the tangent of the angle which the tangent line makes with the axis of abscissas. Now, we have already seen [Art. 21] that the value of this tangent is equal to the derivative of the function y with respect to x.

If, therefore, $y = F(x)$ be the equation of a curve referred to rectangular coördinates,

$$y - y' = \frac{dy'}{dx'}(x - x') \quad (1)$$

will be the equation of a tangent line to the curve at the point y', x'.

NOTE.—In applying this formula the particular coördinates x' and y' must be substituted for x and y in the expression for $\frac{dy}{dx}$.

As an example, let it be required to find the equation of a tangent line to the ellipse.

The equation of the ellipse is

$$a^2 y^2 + b^2 x^2 = a^2 b^2;$$

whence

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y};$$

and the equation of the tangent line is

$$y - y' = -\frac{b^2 x'}{a^2 y'}(x - x').$$

Corollary.—If y be an *implicit* function of x, then, having given the equation $F(x, y) = 0$, we shall have

$$\frac{dy'}{dx'} = -\frac{\dfrac{dF}{dx'}}{\dfrac{dF}{dy'}}$$

and by substituting this value of $\frac{dy'}{dx'}$ in (1), that equation becomes

$$(y - y')\frac{dF}{dy'} + (x - x')\frac{dF}{dx'} = 0 \quad (2),$$

another form for the equation of the tangent line.

92. Problem.— *To find the general differential equation of a normal line to a plane curve.*

The equation of a tangent line being

$$y - y' = a(x - x'),$$

that of the normal will be

$$y - y' = -\frac{1}{a}(x - x').$$

But, by the preceding proposition

$$a = \frac{dy'}{dx'}. \quad \therefore \quad -\frac{1}{a} = -\frac{dx'}{dy'}.$$

Hence, we have for the equation of a normal

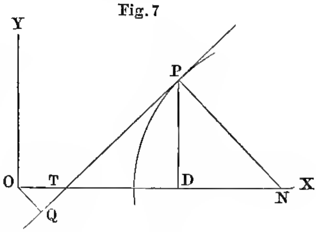
$$y - y' = -\frac{dx'}{dy'}(x - x') \quad (3);$$

and for implicit functions

$$(y - y')\frac{dF}{dx'} - (x - x')\frac{dF}{dy'} = 0 \quad (4).$$

93. Problem. — *To find expressions for the lengths of the tangent, normal, subtangent, subnormal, and perpendicular from the origin upon the tangent.*

Fig. 7



Let O be the origin, OX, OY the axes, PQ the tangent line, PN the normal, PT the tangent for the length of which we are to find an expression, TD the subtangent, ND the subnormal, OQ the perpendicular on the tangent from the origin.

1*st.* To find the subtangent TD. We have $TD = OT - OD$; and making $y = 0$ in (1), the resulting value of x will be the length of OT. Hence,

$$TD = \text{subt} = OT - OD = x - x' = -y'\frac{dx'}{dy'} \quad (a).$$

2*d.* To find the subnormal ND. We have $ND = ON - OD$; and making $y = 0$ in (3), the resulting value of x will be the length of ON. Hence,

$$ND = \text{subnorm} = ON - OD = x - x' = y'\frac{dy'}{dx'} \quad (b).$$

3*d.* To find the length of the tangent. We have

$$\text{tang} = PT = \sqrt{PD^2 + TD^2}$$

$$= \sqrt{y'^2 + y'^2\left(\frac{dx'}{dy'}\right)^2} = y'\sqrt{1 + \left(\frac{dx'}{dy'}\right)^2} \quad (c).$$

4*th.* To find the length of the normal. We have

$$\text{norm} = PN = \sqrt{PD^2 + DN^2}$$

$$= \sqrt{y'^2 + y'^2\left(\frac{dy'}{dx'}\right)^2} = y'\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2} \quad (d).$$

5*th*. To find the length of the perpendicular. We have

$$\text{perp} = OQ = OT \sin OTQ$$

$$= OT \sin PTD = (OD - TD) \sin PTD$$

$$= (OD - TD) \frac{\text{tang } PTD}{\sqrt{1 + \text{tang}^2 PTD}} = \frac{\left\{x' - y' \frac{dx'}{dy'}\right\} \frac{dy'}{dx'}}{\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}}$$

$$= \frac{x' \frac{dy'}{dx'} - y'}{\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}}, \text{ or } \frac{x'dy' - y'dx'}{\sqrt{(dx')^2 + (dy')^2}} \qquad (e).$$

NOTE.—In the expression for the value of the subtangent the negative sign simply indicates that the length of the line is estimated from D toward T. The arithmetical value of the subtangent is that of the second member of (a) without reference to its sign.

94. EXAMPLES.

1. Find the length of the subtangent to the logarithmic curve.

The equation of this curve is $y = a^x$.

$$\therefore \frac{dy}{dx} = a^x \log a = y \log a; \quad \frac{dx}{dy} = \frac{1}{y \log a}.$$

$$\therefore \text{ subt} = -y' \frac{dx'}{dy'} = -\frac{y'}{y' \log a} = -\frac{1}{\log a} = \text{a constant.}$$

2. Find the equations of the tangent and normal, and the lengths of the tangent, subtangent, etc., in all parabolas.

The equation of this class of curves is

$$y^m = a^{m-1} x.$$

Hence, $\dfrac{dy}{dx} = \dfrac{a^{m-1}}{my^{m-1}} = \dfrac{y}{mx}$, and $\dfrac{dx}{dy} = \dfrac{mx}{y}$.

$\therefore$ the equation of

the tangent is $y - y' = \dfrac{y'}{mx'}(x - x')$,

the normal is $y - y' = -\dfrac{mx'}{y'}(x - x')$;

the length of

the subtangent is $-y'\dfrac{dx'}{dy'} = -mx'$,

the subnormal is $y'\dfrac{dy'}{dx'} = \dfrac{y'^2}{mx'}$,

the tangent is $y'\sqrt{1+\left(\dfrac{mx'}{y'}\right)^2} = \sqrt{y'^2 + (mx')^2}$,

the normal is $y'\sqrt{1+\left(\dfrac{y'}{mx'}\right)^2} = \dfrac{y'}{mx'}\sqrt{y'^2 + (mx')^2}$,

the perpendicular $= \dfrac{x'y'(1-m)}{\sqrt{y'^2+(mx')^2}}$.

3. Find the tangent, etc., to the cycloid.

The equation of the cycloid is

$$y = r \operatorname{versin}^{-1}\frac{x}{r} + \sqrt{2rx - x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{r\dfrac{1}{r}}{\sqrt{2\dfrac{x}{r} - \dfrac{x^2}{r^2}}} + \frac{r - x}{\sqrt{2rx - x^2}} = \sqrt{\frac{2r - x}{x}}.$$

Substitute this value of $\frac{dy}{dx}$ in the proper formulas as in the last example.

4. Find the subtangent and subnormal to the cissoid.

The equation of this curve is

$$y^2 = \frac{x^3}{2a - x}.$$

We shall find

$$\text{subtangent} = \frac{x(2a - x)}{3a - x};$$

$$\text{subnormal} = \frac{x^2(3a - x)}{(2a - x)^2}.$$

5. Find the tangent, normal, etc., to the catenary.

The equation of the catenary is

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}.$$

$$\text{Ans. Tangent} = \frac{y^2}{\sqrt{y^2 - c^2}}.$$

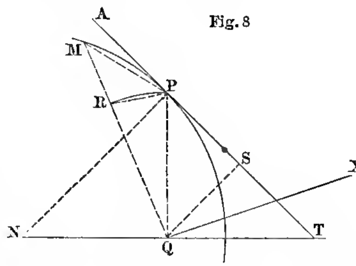
$$\text{Subtangent} = \frac{cy}{\sqrt{y^2 - c^2}}.$$

$$\text{Normal} = \frac{y^2}{c}.$$

$$\text{Subnormal} = \frac{c}{4}\left\{e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}\right\}.$$

95. Problem.—*To find expressions for the length of the tangent, etc., when referred to polar coördinates.*

Fig. 8



Let Q be the pole, and P a point on the curve. Draw the radius-vector QP, the normal PN, and the perpendicular QS. Then, QT being drawn perpendicular to QP, and QX being the fixed axis, we shall have PT the tangent, QT the subtangent, and QN the subnormal.

Designate the radius QP by r, the variable angle by θ, and the angle QPT by μ. Assign to the angle θ an infinitesimal increment $\Delta\theta = PQM$, which will cause r to take an increment Δr.

From Q as a center with radius QP describe an arc PR, and draw the chords MP, PR. Then we shall have arc $PR = r\Delta\theta$, and tang $PMR = \dfrac{\text{chord } PR}{MR}$. (Strictly speaking, this last expression is not rigidly correct, but the terms of the fraction differ by infinitesimals from the true values, and as we are going to take the limits, we may use the one for the other.)

Now $MR = \Delta r$, and the chord PR evidently differs from the arc PR by an infinitesimal. We may, therefore, in accordance with the theory of limits, write

$$\text{tang } PMR = r\frac{\Delta\theta}{\Delta r}.$$

The limit to the angle PMR is QPT or μ. Hence, passing to the limit,

$$\text{tang } \mu = r\frac{d\theta}{dr}.$$

Now, $QT = QP \text{ tang } \mu$; $\therefore$ subtang $= r \times r\dfrac{d\theta}{dr} = r^2 \dfrac{d\theta}{dr}$.

Also, $QN = QP \cot \mu$; $\therefore$ subnormal $= \dfrac{dr}{d\theta}$.

$PT = \sqrt{QP^2 + QT^2}$;

$$\therefore \text{ tang} = \sqrt{r^2 + r^4\left(\frac{d\theta}{dr}\right)^2} = r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}.$$

$PN = \sqrt{QP^2 + QN^2}$; $\therefore$ normal $= \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$.

$QS = \dfrac{QP \times QT}{PT}$;

$$\therefore \text{ perpendicular} = \frac{r^3 \dfrac{d\theta}{dr}}{r\sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \frac{r^2}{\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}}.$$

96. EXAMPLES.

1. *The spiral of Archimedes,* $r = a\theta$.

We have

$$dr = a d\theta; \quad \frac{dr}{d\theta} = a; \quad \frac{d\theta}{dr} = \frac{1}{a}.$$

$$\therefore \text{ subtang} = r^2 \frac{d\theta}{dr} = \frac{r^2}{a}. \quad \text{subnormal} = \frac{dr}{d\theta} = a = \text{const.}$$

$$\text{perp} = \frac{r^2}{\sqrt{r^2 + a^2}}. \quad \text{tang} = ? \quad \text{normal} = ?$$

2. *The logarithmic spiral,* $r = a^\theta$.

We have

$$dr = a^\theta \log a d\theta : \frac{dr}{d\theta} = a^\theta \log a = \frac{a^\theta}{m},$$

representing the modulus of the system by m.

$$\therefore \quad \text{subtang} = rm. \qquad \text{subnorm} = \frac{r}{m}.$$

$$\text{perp} = \frac{r^2}{\sqrt{r^2 + \frac{r^2}{m^2}}} = \frac{rm}{\sqrt{m^2+1}}.$$

$$\text{tang} = ? \qquad \text{normal} = ? \qquad \text{tang}\ \mu = r\frac{d\theta}{dr} = m;$$

$\therefore$ the tangent always makes the same angle with the radius-vector.

3. *The hyperbolic spiral,* $r\theta = a$.

We have $$\frac{d\theta}{dr} = -\frac{a}{r^2}.$$

$$\therefore \quad \text{subtang} = r^2\frac{d\theta}{dr} = -a. \qquad \text{subnorm} = \frac{dr}{d\theta} = -\frac{r^2}{a}.$$

$$\text{tang} = ? \quad \text{normal} = ? \quad \text{perp} = ?$$

4. *The ellipse,* $r = \dfrac{p}{1 + e\cos\theta}$.

We have $r + re\cos\theta = p$; whence

$$dr + e\cos\theta dr - re\sin\theta d\theta = 0, \text{ or } \frac{dr}{d\theta} = \frac{re\sin\theta}{1 + e\cos\theta}.$$

$$\therefore \quad \text{subnormal} = \frac{dr}{d\theta} = \frac{re\sin\theta}{1 + e\cos\theta} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}.$$

$$\text{perp} = \frac{r^2}{\sqrt{r^2 + \frac{r^2e^2\sin^2\theta}{(1+e\cos\theta)^2}}} = \frac{r(1+e\cos\theta)}{\sqrt{1 + 2e\cos\theta + e^2}}$$

$$= \frac{p}{\sqrt{1 + 2e\cos\theta + e^2}}.$$

5. *The lituus,* $r\sqrt{\theta} = a$.

6. *The lemniscata,* $r^2 = 2a^2\cos 2\theta$.

7. *The cardioid,* $r = a(1 - \cos\theta)$.

CHAPTER XIV.

THEORY OF ASYMPTOTES.

97. When two or more lines pass within finite distances of the origin of coördinates, and continually approach each other without ever having contact or intersection, they are said to be **asymptotes** of each other.

98. Let $y = F(x)$, $y_{\prime} = f(x)$, be the equations of two lines referred to the same system of coördinates. Then, if, as x increases, the difference $y_{\prime} - y$ converges continually toward *zero*, the two lines are necessarily asymptotes.

Let $y = D + Ax^a + Bx^b + \ldots\ldots Lx^l$ be the expanded form of $y = F(x)$. Then, in order that the difference $y_{\prime} - y$ may converge toward zero as x increases, the expanded form of $y_{\prime} = f(x)$ must be

$$y_{\prime} = D + Ax^a + Bx^b + \ldots\ldots Lx^l + V,$$

in which V is a quantity which reduces to zero when x is made equal to infinity. For, we have

$$y_{\prime} - y = V = 0 \text{ when } x = \infty.$$

Therefore, *in order that two lines may be asymptotes of each other, the first* m *terms of the development of* y *in their equations must be identical; and the remaining terms must reduce to zero when* x $= \infty$.

99. Rectilinear Asymptotes. — It is obvious that the straight line whose equation is $y = ax + b$ is an asymptote to any curve whose equation can be put under the form $y = ax + b + V$; and we proceed to show how the values of a and b may be found from the equation of the curve itself.

From the equation $y = ax + b + V$, we have

$$\frac{y}{x} = a + \frac{b}{x} + \frac{V}{x}; \text{ and when } x = \infty,$$

$$\frac{y}{x} = a.$$

The value of a in the equation of the asymptote is therefore equal to the limiting value of the ratio $\frac{y}{x}$ found by making $x = \infty$ in the equation of the curve.

Designating this value by k, substituting and transposing, we have

$$b = y - kx - V;$$

and making $x = \infty$, this becomes, since V is then equal to zero,

$$b = \lim (y - kx).$$

We therefore have the following **Rule** for finding the equation of the rectilinear asymptote to a curve.

Find from the equation of the curve the limit to the ratio $\frac{y}{x}$, *and also the limit to the difference* $y - x\lim\left(\frac{y}{x}\right)$, *when* $x = \infty$; *the former of these limits will be the value of* a *in the equation of the asymptote, and the latter will be the value of* b.

In a manner similar to the above we may determine the equation to the asymptote under the form $x = a'y + b'$.

100. In general the ratio $\frac{y}{x}$ will take the indeterminate form $\frac{\infty}{\infty}$ when x is made equal to infinity, and its real value will be found by differentiating the numerator and denomi-

nator, according to the methods already established for indeterminate expressions. We then have, when $x = \infty$,

$$\lim\left(\frac{y}{x}\right) = \frac{dy}{dx},$$

and the value of a is therefore the value which $\frac{dy}{dx}$ assumes when $x = \infty$.

We have also in this case $b = \lim\left(y - \frac{dy}{dx}\,x\right)$, and observing that the equation of a *tangent* line at the point $x'y'$ is

$$y - y' = \frac{dy'}{dx'}\,(x - x'), \text{ or}$$

$$y = \frac{dy'}{dx'}\,x + \left(y' - \frac{dy'}{dx'}\,x'\right),$$

it follows at once that the equation of an asymptote is the same as that of a tangent line when the point of tangency is at an infinite distance from the origin; or, in other words, *the position of the asymptote is the limiting position which that of the tangent approaches as the point of tangency recedes from the origin.*

101. Examples.

1. Find the asymptote to the logarithmic curve $y = e^x$.

Here $a = \lim\left(\frac{y}{x}\right) = \lim \frac{e^x}{x} = 0$ when $x = -\infty$;

$$b = \lim\,(y - ax) = \lim\, y = 0, \text{ when } x = -\infty.$$

$\therefore$ the equation to the asymptote is

$$y = 0, \text{ the equation of the axis of } x.$$

2. The asymptotes to the hyperbola,

$$y = \pm \frac{B}{A} \sqrt{x^2 - A^2}.$$

Here $a = \lim \left(\frac{y}{x} \right) = \lim \left\{ \pm \frac{B}{A} \sqrt{1 - \frac{A^2}{x^2}} \right\} = \pm \frac{B}{A}$
when $x = \infty$:

$$b = \lim (y - ax) = \lim \left(y - \frac{B}{A} x \right) = 0, \text{ when } x = \infty.$$

$\therefore$ the equation to the asymptote is

$$y = \pm \frac{B}{A} x.$$

3. The curve whose equation is $y^3 = cx^2 + x^3$.

Since a is the limit to $\frac{dy}{dx}$ when $x = \infty$, we have

$$a = \lim \left\{ \frac{2cx + 3x^2}{3y^2} \right\} = \frac{2c + 6x}{6y} = \frac{6}{6} = 1 \ :$$

$$b = \lim \left(y - x \frac{dy}{dx} \right) = \lim \left(y - \frac{2cx^2 + 3x^3}{3y^2} \right)$$

$$= \lim \frac{cx^2}{3y^2} = \frac{2cx}{6y} = \frac{2c}{6} = \frac{c}{3} \text{ when } x = \infty.$$

$\therefore$ the equation to the asymptote is

$$y = x + \frac{c}{3}.$$

4. The curve, $\qquad y^2 = \frac{x^3 + cx^2}{x - c}.$

We have

$$\frac{y^2}{x^2}=\frac{x+c}{x-c}=1, \text{ when } x=\infty.$$

$$\therefore \quad a=\lim\left(\frac{y}{x}\right)=\pm 1,$$

$$b=\lim\left(y-ax\right)=\lim\left(y-x\frac{dy}{dx}\right)=\lim\frac{cx^3}{y(x-c)^2}$$

$$=\lim\left\{\frac{\pm cx^2}{(x-c)^2}\right\}=\frac{\pm 2cx}{2(x-c)}=\pm c \text{ when } x=\infty.$$

$\therefore$ the equation of the asymptote is

$$y=\pm(x+c).$$

In this solution we have found a by taking the limit to $\frac{y}{x}$ directly, and b by using $\frac{dy}{dx}$. We have the right to combine the two methods whenever it is convenient to do so.

5. The curve $\qquad xy^2+x^2y=a^3.$

We find

$$\lim\left(\frac{y}{x}\right)=-1, \text{ and } \lim\left(y-x\frac{dy}{dx}\right)=0.$$

$\therefore$ the equation of the asymptote is

$$y=-x.$$

Corollary. — Since $y=\infty$ renders $x=0$, and $x=\infty$ renders $y=0$, it is clear that the two axes are also asymptotes.

6. The curve $\qquad y^2=\frac{x^3}{x-a}.$

Extracting the square root of each member of this equation, we have

$$y = \pm x \left\{ 1 + \frac{a}{2x} + \frac{3a^2}{8x^2} + \text{etc.} \right\}.$$

It is evident from the definition of an asymptote that

$$y = \pm \left(x + \frac{a}{2} \right)$$

is the equation of a rectilinear asymptote to the curve, and also that the equation formed by taking any number of terms of the development will be the equation to a curvilinear asymptote.

This method of determining asymptotes is frequently resorted to when the method illustrated in the preceding examples fails or becomes too cumbrous. The development may be made by any of the usual rules for that purpose.

7. The curve $$y^2 = x^2 \frac{x^2 - 1}{x^2 + 1}.$$

Expanding by division, and extracting the square root, we find $y = \pm x$ to be the equation to the asymptote.

8. The cissoid $$y^2 = \frac{x^3}{2a - x}.$$

It is often possible to discover the existence of an asymptote by a simple inspection of the equation of the curve. Thus, in the present instance, $x = \infty$ renders y imaginary; hence, the curve does not extend indefinitely in the direction of x positive. But $x = 2a$ renders $y = \infty$, and, therefore, $x = 2a$ is the equation of a line which touches the curve at infinity. This line is therefore an asymptote to the curve.

9. The curve $\quad x^2(a^2 - y^2) = a^2y^2.$

Here $x = \infty$ renders $y = \pm a$;

$y = \infty$ renders x imaginary.

$y = \pm (a + h)$ renders x imaginary.

Hence, the curve does not extend in the direction of y beyond the two lines whose equations are $y = \pm a$, and it touches these lines at infinity. The latter are therefore asymptotes to the curve.

10. The curve $\quad (x + a)y^2 = (y + b)x^2.$

It is evident that $x + a = 0$ will give $y = \infty$, and $y + b = 0$ will give $x = \infty$. These are therefore the equations of asymptotes.

By the first method we shall find

$$y = x + b - a$$

as the equation of another asymptote.

In this example we can not put y or x equal to *zero;* this supposition would render $x + a = \infty$, $y + b = \infty$, either of which equations would render the two members of the given equation inconsistent with each other.

102. There is no *general* method by which the values of a and b in the equation of the asymptote can be found in all cases; but whenever the equation of the curve can be decomposed into several parts, each of which is a *homogeneous* function of the variables, the values of a and b may be determined by the following simple process:

Let m, n, etc., be the degrees of the different homogeneous functions, so that we shall have for the equation of the curve

$$x^m F\left(\frac{y}{x}\right) + x^n f\left(\frac{y}{x}\right) + \cdots\cdots = 0, \quad m > n.$$

Representing $\frac{y}{x}$ by s and dividing by x^m, we have

$$F(s) + \frac{1}{x^{m-n}} f(s) + \ldots\ldots = 0;$$

whence, making $x = \infty$,

$$F(s) = 0, \text{ or } F\left(\frac{y}{x}\right) = 0,$$

and the value of a is evidently to be found among the real roots of this equation.

To find the value of b corresponding to any real value of a, put $y = ax + t$, whence $\frac{y}{x} = a + \frac{t}{x}$, and the equation of the curve becomes

$$x^m F\left(a + \frac{t}{x}\right) + x^n f\left(a + \frac{t}{x}\right) + \ldots\ldots = 0.$$

But since $F(a) = F\left(\frac{y}{x}\right) = 0$, we have [Art. 46]

$$F\left(a + \frac{t}{x}\right) = \frac{t}{x} F'\left(a + \theta \frac{t}{x}\right); \text{ whence}$$

$$x^{m-1} t F'\left(a + \theta \frac{t}{x}\right) + x^n f\left(a + \frac{t}{x}\right) + \ldots\ldots = 0, \text{ or}$$

$$t F'\left(a + \theta \frac{t}{x}\right) + \frac{1}{x^{m-n-1}} f\left(a + \frac{t}{x}\right) = 0.$$

If, now, in this equation we suppose $x = \infty$, which renders $t = b$, and if $F'(a)$, $f(a)$, have finite values, then we shall have

1st, for $n < m - 1$, $\lim(t) = b = 0$;

2d, for $n = m - 1$, $\lim(t) = b = -\frac{f(a)}{F'(a)}$;

3d, for $n > m - 1$, $\lim(t) = b = \infty$, and there is no asymptote.

$\therefore$ in the first case the equation of the asymptote is

$$y = ax;$$

in the second case the equation of the asymptote is

$$y = ax - \frac{f(a)}{F'(a)}.$$

In the first case, in which $n < m - 1$, the equations $y = ax$, $F(a) = 0$, give $F\left(\frac{y}{x}\right) = 0$, and consequently $x^m F\left(\frac{y}{x}\right) = 0$, which is also the equation of asymptotes: so that whenever $n < m - 1$, or $m > n + 1$, the function of the degree m placed equal to zero gives us the equation of the asymptotes.

Examples.

1. The curve $Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$.

We have

$$x^2 \left\{A\frac{y^2}{x^2} + B\frac{y}{x} + C\right\} + x\left\{D\frac{y}{x} + E\right\} + F = 0.$$

$$\therefore\ F\left(\frac{y}{x}\right) = F(a) = Aa^2 + Ba + C = 0, \text{ and}$$

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A};$$

$$f\left(\frac{y}{x}\right) = f(a) = Da + E; \quad -\frac{f(a)}{F'(a)} = b = -\frac{Da + E}{2Aa + B};$$

and the equation of the asymptote is

$$y = ax - \frac{Da + E}{2Aa + B}.$$

The values of a are imagiuary if $B^2 - 4AC < 0$, and the value of b becomes infinite if $B^2 - 4AC = 0$.

The asymptote is real only when $B^2 - 4AC > 0$.

This example is the solution of the problem of asymptotes as applied to the *conic sections.*

2. The folium of Descartes, $y^3 + x^3 - 3cxy = 0$.

We have

$$x^3\left\{\frac{y^3}{x^3} + 1\right\} - x^2\left\{3c\frac{y}{x}\right\} = 0.$$

Hence, $F(a) = a^3 + 1 = 0$, and $a = -1$:

$$-\frac{f(a)}{F'(a)} = \frac{3ca}{3a^2} = \frac{c}{a} = -c.$$

$\therefore$ the equation of the asymptote is

$$y = -x - c.$$

3. The curve $\qquad y^3 - x^3 - cx^2 = 0.$

We have

$$x^3\left\{\frac{y^3}{x^3} - 1\right\} - x^2\{c\} = 0.$$

Hence, $F(a) = a^3 - 1 = 0$, and $a = 1$;

$$-\frac{fa}{F'(a)} = \frac{c}{3a^2} = \frac{c}{3}.$$

$\therefore$ the equation of the asymptote is

$$y = x + \frac{c}{3}.$$

4. The curve $y^2(Ay + Bx) = A^2y^2 + B^2x^2$.

Placing this under the form

$$x^3\left\{A\frac{y^3}{x^3} + B\frac{y^2}{x^2}\right\} - x^2\left\{A^2\frac{y^2}{x^2} + B^2\right\} = 0,$$

we find for the equation of the asymptote

$$y = -\frac{B}{A}x + 2A.$$

5. The curve $y^4 - x^4 + 2cx^2y = 0$.

Placing this under the form

$$x^4\left\{\frac{y^4}{x^4} - 1\right\} + x^3\left\{2c\frac{y}{x}\right\} = 0,$$

we have for the equation of the asymptote

$$y = \pm x - \frac{c}{2}.$$

6. The curve $Ay^3 - Bx^3 + Cxy = 0$.

The equation of the asymptote is

$$y = x\sqrt[3]{\frac{B}{A}} - \frac{C}{3\sqrt[3]{A^2B}}.$$

103. If the equation of a curve be given in terms of polar coördinates, it is evident that the curve will have a rectilinear asymptote whenever for an infinite value of the radius-vector the perpendicular on the tangent is finite.

In order, then, to determine whether there is an **asymptote to a polar curve,** we must find from the equation of the curve all those values of θ which render r infinite, and by substitution in the expression for the perpendicular, find whether the length of this line is finite.

EXAMPLES.

1. The hyperbolic spiral, $r\theta = a$.

We have

$$r = \infty \text{ when } \theta = 0; \text{ also } \frac{dr}{d\theta} = -\frac{a}{\theta^2}.$$

$$\therefore \text{ perp} = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = \frac{r^2\theta^2}{\sqrt{r^2\theta^4 + a^2}} = \frac{a^2}{a} = a,$$

when $\theta = 0$.

Hence, this curve has an asymptote parallel to the axis, and at a distance a from it.

2. The lituus $\qquad r^2\theta = a^2$.

We have

$$r = \infty \text{ when } \theta = 0; \text{ also } \frac{dr}{d\theta} = -\frac{a}{2\theta\sqrt{\theta}}.$$

$$\therefore \text{ perp} = \frac{r^2}{\sqrt{r^2 + \frac{a^2}{4\theta^3}}} = \frac{2r^2\theta}{\sqrt{4\theta(r^2\theta) + r^2}} = \frac{2a^2}{r} = 0$$

when $r = \infty$.

Hence, the axis is an asymptote to this curve.

104. Circular Asymptotes.—When the polar equation of a curve is of such a form that when $\theta = \infty$ the value of r is finite and equal to a, the curve will make an infinite number of convolutions about the pole before reaching the circumference of a circle whose center is the pole, and whose radius is a. This circle is then an asymptote to the curve; it will be exterior to the curve if every finite value of r is less than a, and if $r > a$ the curve will be exterior to the circle.

As an example, let us take the curve

$$r = \frac{a\theta^2}{\theta^2 - 1}.$$

This may be put under the form

$$r = \frac{a}{1 - \frac{1}{\theta^2}},$$

and if $\theta = \infty$ we have $r = a$.

Every finite value of θ will render $r > a$, and therefore the curve is exterior to the circle.

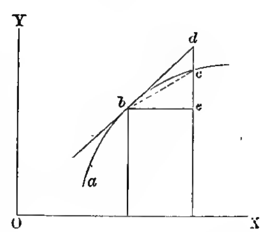
We shall also find two rectilinear asymptotes to this curve.

CHAPTER XV.

DIFFERENTIALS OF THE ARC, AREA, AND INCLINATION OF A PLANE CURVE. CONVEXITY AND CONCAVITY.

105. Before discussing the subjects of this chapter it is necessary to demonstrate that *the limit to the ratio of an arc and its chord is unity.*

Fig. 9



Let $y = F(x)$ be the equation of the curve abc, and let x, y, be the coördinates of the point b. Take another point, c, so near the former that the chord bc, and all chords drawn from b to points between b and c, shall lie on the same side of their corresponding arcs. Then we may assume that the arc bc is contained between the chord bc and the tangent bd, and the proposition

above stated will be proved if we can show that the limit to the ratio of the *chord* and *tangent* is unity. Now, we have

$$bd^2 - bc^2 = de^2 - ce^2;$$

and the limit to the *ratio* of de and ce being evidently unity, the limit to their *difference* is [Art. 14] *zero.*

$$\therefore \quad \lim (bd^2 - bc^2) = 0;$$

$$\therefore \quad \lim \left(\frac{bd}{bc}\right)^2 = 1;$$

$$\therefore \quad \lim \frac{bd}{bc} = 1, \quad \text{and} \lim \left(\frac{bc}{bd}\right) = 1.$$

The limit to the ratio of the chord and tangent being unity, and the arc being always between these two, it follows that the limit to the ratio of the *chord* and *arc* is unity, and, therefore, the chord may be substituted for the arc in any ratio or series whose limit is to be found.

106. Problem.—*To find an expression for the differential of an arc of a plane curve.*

Designating the chord bc (Fig. 9) by Δch, be by Δx, ce by Δy, we have

$$\Delta ch = \sqrt{(\Delta x)^2 + (\Delta y)^2}; \text{ whence } \frac{\Delta ch}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Passing to the limits, and observing that the limit to $\dfrac{\Delta ch}{\Delta x}$ is the same as that to $\dfrac{\Delta \text{ arc}}{\Delta x}$, we have

$$\frac{d \text{ arc}}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ or}$$

$$d \text{ arc} = \sqrt{dx^2 + dy^2} \qquad (1).$$

Corollary.—If the curve be referred to *polar coördinates*, we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

$$\therefore \; dx = \cos \theta dr - r \sin \theta d\theta;$$

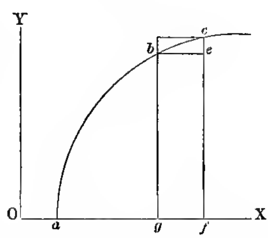
$$dy = \sin \theta dr + r \cos \theta d\theta.$$

Substituting these values in (1), and reducing, we have

$$d \text{ arc} = \sqrt{r^2 d\theta^2 + dr^2} \qquad (2).$$

107. Problem.—*To find an expression for the differential of the* area *of a plane curve, referred to rectangular axes.*

Fig. 10



Let abg be an area included between the arc ab, the ordinate bg, and the abscissa ag. It is required to find the limit* to the area $bgfc$ corresponding to the increment gf of the abscissa. Completing the rectangles bf and cg, the area $bgfc$ is comprised between these two rectangles, and the limit to their ratio is evidently equal to the limit to the ratio of either of them to the area $bgfc$.

Now, since these two rectangles have the same base, the limit to their ratio is equal to the limit to the ratio of their altitudes, and this limit is evidently *unity.*

Hence, the limit to the ratio of either of the rectangles to $bgfc$ is unity.

$$\text{Now, } gfeb = y\Delta x.$$

*In this, and in several succeeding propositions, we mean by the term limit the quantity which can replace a variable in a ratio or series whose limit we wish to find. As the word will also be used frequently in its ordinary sense, the student must carefully distinguish between the two meanings, when they occur in the same demonstration.

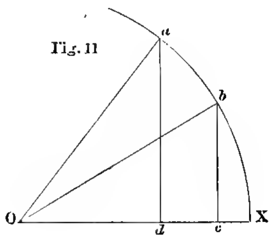
$$\therefore \frac{bgfc}{gfeb} = \frac{\Delta \text{ area}}{y\Delta x}; \quad \text{and } \lim \frac{bgfc}{gfeb} = \lim \frac{\Delta \text{ area}}{y\Delta x}$$

$$= \frac{d \text{ area}}{ydx} = 1.$$

$$\therefore \quad d \text{ area} = ydx \qquad (3).$$

108. Problem.—*To find an expression for the differential of an area when referred to polar coördinates.*

Fig. 11



The problem to be solved is to find the limit to the area Oab included between the two radii Oa, Ob, and the arc ab. Let O be the origin, OX the fixed axis, let x, y, be the rectangular coördinates of a, and $x + \Delta x$, $y + \Delta y$, those of b.

Then we shall have

$$Oab = \Delta \text{ area} = dcba + aOd - Obc:$$

$$\therefore \lim \frac{\Delta \text{ area}}{\Delta x} = \lim \frac{dcba}{\Delta x} + \lim \frac{aOd}{\Delta x} - \lim \frac{Obc}{\Delta x}.$$

Now, $\lim \frac{dcba}{\Delta x} = \lim \frac{y\Delta x}{\Delta x} = y$; $\lim \frac{aOd}{\Delta x} = \lim \frac{xy}{2\Delta x} = \frac{xy}{2dx}$;

$$\lim \frac{Obc}{\Delta x} = \lim \frac{1}{2}\left\{\frac{(x+\Delta x)(y+\Delta y)}{\Delta x}\right\} = \frac{1}{2}\frac{xy}{dx} + \frac{xdy}{2dx} + \frac{1}{2}y.$$

$$\therefore \lim \frac{\Delta \text{ area}}{\Delta x} = \frac{d \text{ area}}{dx}$$

$$= y + \frac{1}{2}\frac{xy}{dx} - \frac{1}{2}\frac{xy}{dx} - \frac{xdy}{2dx} - \frac{1}{2}y$$

$$= \frac{1}{2}\left\{y - x\frac{dy}{dx}\right\}; \text{ and}$$

$$d \text{ area} = \frac{1}{2}(ydx - xdy).$$

Finally, substituting for x and y their values, $r \cos \theta$, $r \sin \theta$, and differentiating, we have

$$d \text{ area} = \frac{1}{2} r^2 d\theta \qquad (4).$$

109. Problem. — *To find an expression for the differential of the angle which a tangent line to a curve makes with the axis of abscissas.*

Designating this angle by ϕ, we have

$$\text{tang } \phi = \frac{dy}{dx}.$$

Hence, by differentiation,

$$\sec^2 \phi d\phi = \frac{d^2y}{dx^2} dx;$$

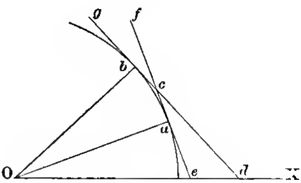
$$\therefore \quad d\phi = \frac{\frac{d^2y}{dx^2} dx}{\sec^2 \phi} = \frac{\frac{d^2y}{dx^2} dx}{1 + \text{tang}^2 \phi} = \frac{\frac{d^2y}{dx^2} dx}{1 + \left(\frac{dy}{dx}\right)^2} \qquad (5).$$

This value of $d\phi$ is called the **angle of contact** of the curve and tangent. It may be considered as the limit of $\Delta\phi$, which is the angle made by two lines which are tangent to the curve at points whose abscissas differ from each other by Δx; and in all cases where ϕ is the independent variable, $d\phi$ may be taken as equal to $\Delta\phi$.

110. Problem.—*To find the angle of contact when the curve is referred to polar coördinates.*

This angle is the limit to the angle fcg or ecd made by two tangents fe and gd.

Fig. 12



Now, we have

$$ecd = cdx - cex;$$

$$cdx = Obd + bOx;$$

$$cex = Oae + aOx.$$

$$\therefore\ cdx - cex = Obd - Oae + bOx - aOx = Obd - Oae + bOa.$$

Again, $Obd - Oae = \Delta Oae$, and $bOa = \Delta\theta$.

$$\therefore\ ecd = \Delta\phi = \Delta Oae + \Delta\theta,$$

$$\lim\frac{\Delta\phi}{\Delta\theta} = \lim\frac{\Delta Oae}{\Delta\theta} + \lim\frac{\Delta\theta}{\Delta\theta},\ \text{or}$$

$$\frac{d\phi}{d\theta} = \frac{dOae}{d\theta} + 1,\ \text{and}$$

$$d\phi = dOae + d\theta \qquad (a).$$

Now, $Oae = \mu$ [Art. 95], and $\tan \mu = r\dfrac{d\theta}{dr} = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$.

$$\therefore\ \sec^2\mu d\mu = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}d\theta;\ \text{and}$$

$$d\mu = dOae = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2\left\{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}}d\theta = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}d\theta.$$

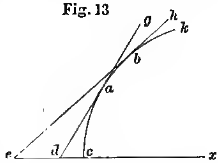
Substituting this value of $dOae$ in (a), we have

$$d\phi = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \qquad (6).$$

111. Concavity and Convexity.—A curve is said to be **concave** to a given straight line, at any of its points, when the two parts of the curve immediately adjacent to such point are contained between the straight line and the tangent at that point. It is **convex** to the straight line when the tangent lies between the curve and the straight line.

Let $cabk$ be a curve concave to the axis of x, and let gad, hbe, be two tangents drawn at points a and b, infinitesimally near each other.

Fig. 13



It is evident that the angle hex is less then gdx, and that, in passing from a to b, ϕ is a *decreasing* function. If, then, x be the abscissa of a the sign of $\frac{d\phi}{dx}$ must be *negative* [Art. 18], and, therefore,

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \text{a negative quantity.}$$

The denominator of this fraction being essentially positive, the sign of the fraction will be the same as that of its numerator.

Hence, *for a point where the curve is concave to the axis of* x $\frac{d^2y}{dx^2}$ *must be negative;* and in the same way we may show that *for a point where the curve is convex to the axis of* x, $\frac{d^2y}{dx^2}$ *must be positive.*

It is obvious that these conditions will be reversed when the point under consideration is below the axis of x; and combining the two cases, we see that the general condition for concavity is that y and $\frac{d^2y}{dx^2}$ shall have opposite signs, and for convexity the condition is that they shall have the same sign.

If the curve be referred to polar coördinates, it is easy to see that if a perpendicular p be drawn from the pole to the tangent, then, in any portion of the curve which is concave to the axis, p will increase as r increases; while for a convex portion, p will decrease as r increases.

Hence, for *concavity*, $\frac{dp}{dr}$ must be a *positive* quantity; and for *convexity*, $\frac{dp}{dr}$ must be a *negative* quantity.

The analytical expression for $\frac{dp}{dr}$ may be readily deduced from the equation already given in Art. 95;

$$p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}.$$

We have

$$\frac{dp}{dr} = \frac{2r\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} - r^2 \frac{\left\{ r + \frac{d^2r}{d\theta^2} \right\}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{r^3 + 2r\left(\frac{dr}{d\theta}\right)^2 - r^2\frac{d^2r}{d\theta}}{\left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{3}{2}}} = \frac{r\,\frac{d\phi}{d\theta}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}.$$

It follows from this result, that if r and $\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$ be *positive*, the sign of $\frac{dp}{dr}$ will be the same as that of $\frac{d\phi}{d\theta}$, and this will evidently depend upon the sign of the numerator of $d\phi$ in formula (6) of this chapter.

In applying this formula, it is necessary to consider θ as always positive.

CHAPTER XVI.

CURVATURE AND CONTACT OF CURVES; EVOLUTES AND INVOLUTES.

112. Curvature.—*The curvature of any arc of a plane curve is the external angle contained between the tangents drawn through its extremities.*

If in every curve this angle varied directly as the length of the arc, as is the case in the circle, we might obtain the curvature of a unit of arc by dividing that of the whole arc by the length of the arc. But in every case except the circle, because of the variation of curvature, the quotient so obtained will be only the mean or *average* curvature of a unit. If, now, beginning at any point of a curve, we take an arc of arbitrary but infinitesimal length, its mean curvature will evidently vary as the arc tends toward zero, and it will tend toward a certain limit, which is in general determinable, and is called *the curvature of the arc at the given point.*

Let Δs and $\Delta\phi$ be the arc and the angle between the tangents drawn through its extremities. Then $\frac{\Delta\phi}{\Delta s}$ will be the mean curvature of the unit of the arc, and

$$\lim \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$$

will be the expression for the curvature at the point at which the arc begins.

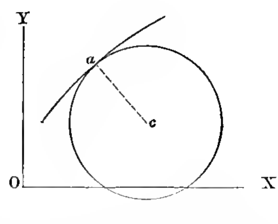
113. The curvature of a circle being uniform, we shall have for the curvature of any point,

$$\lim \frac{\Delta \phi}{\Delta s} = \frac{\Delta \phi}{\Delta s};$$

or, since $\Delta s = R\Delta \phi$, R being the radius,

$$\frac{\Delta \phi}{\Delta s} = \frac{\Delta \phi}{R\Delta \phi} = \frac{1}{R} = \text{curvature of a circle at any point.}$$

Fig. 14



Now, whatever be the curvature at any point of a plane curve, there is obviously a circle which has the same curvature, and this circle can be placed tangent to the curve at that point, with its radius coinciding in direction with the normal to the curve at that point. This circle is called the **circle of curvature** of that point of the curve; its center is the **center of curvature,** and its radius is the **radius of curvature** of the given point.

114. Problem.—*To find an expression for the radius of curvature of any point of a plane curve.*

We have

Curvature of curve = curvature of circle, or

$$\frac{d\phi}{ds} = \frac{1}{R}; \quad \text{whence } R = \frac{ds}{d\phi}.$$

Substituting in this the values of ds and $d\phi$, viz:

$$ds = \sqrt{dx^2 + dy^2}, \text{ and } d\phi = \frac{\frac{d^2y}{dx^2}\,dx}{1 + \left(\frac{dy}{dx}\right)^2},$$

it becomes

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad (7);$$

and substituting the values of ds and $d\phi$ in terms of polar coördinates, viz: $x = r\cos\theta$, $y = r\sin\theta$, we have

$$R = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \qquad (8).$$

115. Problem.— *To determine the radius of curvature in terms of a new variable,* t.

Applying the formulas for changing the independent variable [Art. 85], we have, at once,

$$R = \frac{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}, \text{ or}$$

$$R = \frac{ds^3}{dx\,d^2y - dy\,d^2x} \qquad (9).$$

Corollary 1.—If the new variable be s, we have

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} = 1; \quad \frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0.$$

$$\therefore \quad \frac{d^2x}{ds^2} = -\frac{dy}{ds}\frac{d^2y}{ds^2}\frac{ds}{dx},$$

and substituting in the above value of R, we have

$$R = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}} = \frac{\sqrt{1-\left(\frac{dy}{ds}\right)^2}}{\frac{d^2y}{ds^2}} \qquad (10).$$

Corollary 2.—If the curve be referred to polar coördinates, and we make p the independent variable, we have at once, by comparing formula (8) with the value of $\frac{dp}{dr}$ in Art. 111,

$$R = r\frac{dr}{dp} \qquad (11).$$

116. Contact of Curves.—Let $Y = F(x)$, $y = f(x)$ be the equations of two curves referred to the same axes. Giving to x an infinitesimal increment, h, and expanding by Taylor's formula, we have

$$y_{\prime} = y + \left(\frac{dy}{dx}\right)\frac{h}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.\,2} + \text{etc.} \qquad (1).$$

$$Y_{\prime} = Y + \left(\frac{dY}{dx}\right)\frac{h}{1} + \left(\frac{d^2Y}{dx^2}\right)\frac{h^2}{1.\,2} + \text{etc.} \qquad (2).$$

If, now, in the equations of these two curves we have for any given value of x, $Y = y$, the two curves will evidently have a *common point*.

If, also, $\frac{dY}{dx} = \frac{dy}{dx}$, they will have a *common tangent;* and if, at the same time, $\frac{d^2Y}{dx^2} = \frac{d^2y}{dx^2}$, they will, after passing the common point, diverge from each other less than if

$$\frac{d^2Y}{dx^2} \gtrless \frac{d^2y}{dx^2}.$$

Two curves, whose *first derivatives* only are the same, for the same values of y and x, are said to have contact of the first order; and, generally, if the first n derivatives are the same, the curves have **contact of the n^{th} order.**

117. The order of contact which one curve, which is given by means of its general equation, may have with another entirely given, depends on the number of arbitrary constants in the equation of the first curve.

Contact of the *first* order requires *two* conditions, viz:

$$Y = y, \text{ and } \frac{dY}{dx} = \frac{dy}{dx};$$

hence, the equation of the first curve must contain two arbitrary constants by means of which these conditions may be determined.

Contact of the second order requires *three* conditions, viz:

$$Y = y, \quad \frac{dY}{dx} = \frac{dy}{dx}, \quad \frac{d^2Y}{dx^2} = \frac{d^2y}{dx^2};$$

hence, the equation of the first curve must contain *three* constants, and so on. The general equation of the straight line contains but two arbitrary constants; hence, *in general*, the straight line can not have contact of a higher order than the first with any plane curve.

The general equation of the circle has but three arbitrary constants; hence, *in general*, the highest order of contact possible between a circle and another curve is the second. We say in general, because for certain particular points in a curve the order of contact between it and the circle may be of a higher degree than the second.

The circle which has contact of the second order with another curve at a given point is called the **osculatory circle** of that point.

118. Problem.— *To determine the radius of the osculatory circle of any point of a plane curve.*

The formula for the radius of curvature being

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

we shall have the required radius by substituting in this formula the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, taken from the equation of the circle. But since the circle has contact of the second order with the given curve, the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, are the same for the circle and for the curve. Hence, the radius of the required circle may also be found by taking the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, from the equation of the curve. The two results being identical, it follows that *the radius of the osculatory circle at any given point of a curve is equal to the radius of curvature of the curve at that point;* and, consequently, *the osculatory circle is the circle of curvature, and its center is the center of curvature.*

Corollary.—If we differentiate the value of R with respect to x, and place the first derivative equal to zero, we shall have one of the conditions for a maximum or minimum value of R. Performing this operation we have

$$3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = 0 \qquad (1).$$

If, now, we take as the equation of the osculatory circle

$$(x - a)^2 + (y - b)^2 = R^2,$$

and differentiate this three times, we obtain, after reduction,

$$3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2-\frac{d^3y}{dx^3}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}=0 \qquad (2),$$

an equation of the same form with (1).

Now, since $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, taken from the equation of the circle, are equal to the same quantities taken from the equation of the curve, it follows that $\frac{d^3y}{dx^3}$ must be the same in both equations. But this, in connection with the foregoing, is the condition that the two curves shall have contact of the *third* order. Hence it follows that *at those points for which the value of the radius of curvature is either a maximum or minimum, the circle and curve have contact of the third order.*

119. Evolutes. — The curve which is the locus of the centers of all the osculatory circles of a given curve, is called the **evolute** of that curve, and the given curve is called the **involute** of its evolute.

120. Problem. — *To find the equation of the evolute of a plane curve.*

Let $y=F(x)$ be the equation of the curve, and let

$$(x-a)^2+(y-b)^2=R^2$$

be the equation of the osculatory circle.

Differentiating this equation, we obtain

$$(x-a)+(y-b)\frac{dy}{dx}=0,$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}=0,$$

in which the derivatives are the same as those taken from the equation of the curve.

If we combine these equations with that of the curve so as to eliminate x and y, the resulting equation will be a relation between a and b, the coördinates of the center of the osculatory circle, and it will therefore be the required equation of the evolute.

121. If we differentiate the equation

$$(x-a)+(y-b)\frac{dy}{dx}=0$$

with reference to x, considering all the quantities as variable, we shall have

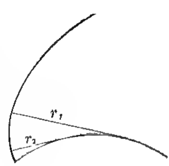
$$1+\left(\frac{dy}{dx}\right)^2-\frac{da}{dx}+(y-b)\frac{d^2y}{dx^2}-\frac{db}{dx}\frac{dy}{dx}=0;$$

But
$$1+\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}=0;$$

$$\therefore\quad -\frac{da}{dx}=\frac{db}{dx}\frac{dy}{dx}, \text{ and } \frac{dy}{dx}=-\frac{da}{db}.$$

Now $\dfrac{db}{da}$ is the tangent of the angle which the tangent line to the evolute makes with the axis of x, and $\dfrac{dy}{dx}$ is the tangent of the angle which the tangent line to the curve makes with the axis of x. The above equation indicates that these two lines are perpendicular to each other, and since a line passing through the point of tangency, perpendicular to the tangent, is the normal, it follows that *the tangent to the evolute is the normal to the involute.*

Fig. 15



122. Resuming the equation

$$(x-a)^2+(y-b)^2=R^2,$$

and differentiating with respect to a, we have

$$(x-a)\left(\frac{dx}{da}-1\right)+(y-b)\left(\frac{dy}{da}-\frac{db}{da}\right)=R\,\frac{dR}{da}.$$

But

$$(x-a)\frac{dx}{da}+(y-b)\frac{dy}{da}=\frac{dx}{da}\left\{(x-a)+(y-b)\frac{dy}{dx}\right\}=0.$$

$$\therefore\;\; x-a+(y-b)\frac{db}{da}=-R\,\frac{dR}{da}.$$

Now $$\frac{db}{da}=-\frac{dx}{dy}\text{ (Art. 121)}=\frac{y-b}{x-a}.$$

$\therefore$ by substitution in the last equation,

$$(x-a)\left\{1+\left(\frac{db}{da}\right)^2\right\}=-R\,\frac{dR}{da} \qquad (1).$$

Again,

$$(x-a)^2+(y-b)^2=(x-a)^2\left\{1+\left(\frac{y-b}{x-a}\right)^2\right\}$$

$$=(x-a)^2\left\{1+\left(\frac{db}{da}\right)^2\right\}=R^2;$$

$$\therefore\;\; (x-a)\left\{1+\left(\frac{db}{da}\right)^2\right\}^{\frac{1}{2}}=-R \qquad (2).$$

From (1) and (2) we have

$$\left\{1+\left(\frac{db}{da}\right)^2\right\}^{\frac{1}{2}}=\frac{dR}{da};\quad \therefore\;\; dR=\sqrt{db^2+da^2}$$
$$=d\text{ (arc of evolute)}.$$

Now we have seen (Art. 47, Cor. 2) that when two variables have the same derivative with respect to another

variable, they differ from each other by a constant. Therefore, designating the arc of the evolute by S, we have

$$R = S + c.$$

If R', R'', S', S'', be two radii of curvature and the corresponding arcs of the evolute, we shall have

$$R' = S' + c, \text{ and } R'' = S'' + c.$$

$$\therefore \quad R' - R'' = S' - S'';$$

or the difference between two radii of curvature is equal to the arc of the evolute intercepted between them.

123. From the foregoing properties of the evolute it is easy to see that if a string be wound round the evolute of a curve, and then be unwound, keeping the part unwound always tangent to the curve, the end of the string will describe the involute, and every point in the string will describe a curve similar to the involute.

124. Problem.—*To find the equation of the involute to a curve.*

If we eliminate a and b between the following equations:

$$F(a, b) = 0, \text{ the equation of the evolute;}$$

$$\frac{da}{db} = -\frac{dy}{dx}; \quad (x - a) + (y - b)\frac{dy}{dx} = 0;$$

there will result a relation between x and y, which will be the equation of the involute. This will generally be a *differential* equation.

125. Examples.

1. Find the radius of curvature of all the conic sections.

The equation of these curves is

$$y^2 = mx + nx^2.$$

$$\therefore\ \frac{dy}{dx} = \frac{m + 2nx}{2(mx + nx^2)^{\frac{1}{2}}};\quad \frac{d^2y}{dx^2} = \frac{m^2}{4(mx + nx^2)^{\frac{3}{2}}} = \frac{m^2}{4y^3}.$$

Now since

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{(\text{normal})^3}{y^3}\ \text{[Art. 93]},$$

we have

$$R = \frac{(\text{normal})^3}{y^3\,\frac{d^2y}{dx^2}} = \frac{(\text{normal})^3}{\frac{1}{4}m^2} = \frac{(\text{normal})^3}{(\text{semi-parameter})^2}.$$

2. The radius of curvature of each of the conic sections.

3. The logarithmic curve $y = a^x$.

We have

$$\frac{dy}{dx} = a^x \log a = \frac{y}{m};\quad \frac{d^2y}{dx^2} = \frac{1}{m}\,\frac{dy}{dx} = \frac{y}{m^2}.$$

$$\therefore\ R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(m^2 + y^2)^{\frac{3}{2}}}{my}.$$

4. The cycloid.

We have already found [Art. 94, Ex. 3]

$$\frac{dy}{dx} = \sqrt{\frac{2r - x}{x}}.\quad \therefore\ \frac{d^2y}{dx^2} = -\frac{r}{x\sqrt{2rx - x^2}}:$$

$$R = \frac{\left\{1 + \frac{2r - x}{x}\right\}^{\frac{3}{2}}}{-\frac{r}{x\sqrt{2rx - x^2}}} = -2\sqrt{2r(2r - x)}$$

$= \text{twice the } \textit{normal}.$

5. The logarithmic spiral $r = a^\theta$.

We have

$$\frac{dr}{d\theta} = a^\theta \log a = \frac{r}{m}; \qquad \frac{d^2r}{d\theta^2} = \frac{r}{m^2}.$$

$$\therefore\ R = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left\{r^2 + \frac{r^2}{m^2}\right\}^{\frac{3}{2}}}{r^2 + \frac{r^2}{m^2}} = \left\{r^2 + \frac{r^2}{m^2}\right\}^{\frac{1}{2}}$$

$= \text{normal.}$

6. The hyperbolic spiral $r\theta = a$.

We have

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2} = -\frac{r^2}{a}: \quad \frac{d^2r}{d\theta^2} = \frac{2a}{\theta^3} = \frac{2r^3}{a^2}.$$

$$\therefore\ R = \frac{r(a^2 + r^2)^{\frac{3}{2}}}{a^3}.$$

7. The catenary,

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}. \qquad \text{Ans. } R = -\frac{y^2}{c}.$$

8. The evolute of the parabola $y^2 = 2px$.

We have

$$\frac{dy}{dx} = \frac{p}{y}; \quad \frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

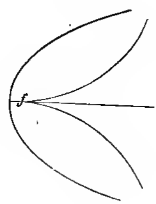
These values, substituted in the differential equations of the circle, give

$$x-a=-\frac{y^2}{p}-p=-2x-p;\ \text{whence}\ x=\frac{a-p}{3}:$$

$$y-b=\frac{y^3}{p^2}+y;\ \text{whence}\ y^3=-bp^2;\ y^2=b^{\frac{2}{3}}p^{\frac{4}{3}}.$$

These values, substituted in the equation of the parabola, give

Fig. 16



$$b^{\frac{2}{3}}=\frac{2}{3p^{\frac{1}{3}}}(a-p),\ \text{or}$$

$$b^2=\frac{8}{27p}(a-p)^3,$$

the equation of the evolute, which is the *semi-cubical parabola.*

9. The evolute of the ellipse $A^2y^2+B^2x^2=A^2B^2$.

We have

$$\frac{dy}{dx}=-\frac{B^2x}{A^2y};\quad \frac{d^2y}{dx^2}=-\frac{B^4}{A^2y^3}.$$

Substituting these values in the differential equations of the circle, we obtain

$$x-a=\frac{x(A^4y^2+B^4x^2)}{A^4B^2}=\frac{x(A^4B^2-A^2B^2x^2+B^4x^2)}{A^4B^2}:$$

$$\therefore\ x^3=\frac{A^4a}{A^2-B^2}\ \text{and}\ x^2=\frac{A^{\frac{8}{3}}a^{\frac{2}{3}}}{(A^2-B^2)^{\frac{2}{3}}}.$$

$$y - b = \frac{y(A^4y^2 + B^4x^2)}{A^2B^4} = \frac{y(A^2B^4 - A^2B^2y^2 + A^4y^2)}{A^2B^4}:$$

$$\therefore\ y^3 = \frac{B^4b}{A^2 - B^2},\ \text{and}\ y^2 = \frac{B^{\frac{8}{3}}b^{\frac{2}{3}}}{(A^2 - B^2)^{\frac{2}{3}}}.$$

These values of x^2 and y^2, substituted in the equation of the ellipse, give, for that of the evolute,

$$A^{\frac{2}{3}}a^{\frac{2}{3}} + B^{\frac{2}{3}}b^{\frac{2}{3}} = (A^2 - B^2)^{\frac{2}{3}}.$$

10. The evolute of the cycloid.

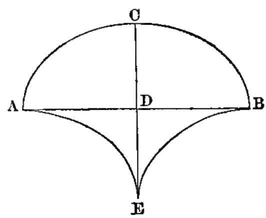
The equation of the cycloid is

$$y = r\ \text{versin}^{-1}\frac{x}{r} + \sqrt{2rx - x^2}.$$

$$\therefore\ \frac{dy}{dx} = \sqrt{\frac{2r - x}{x}};\quad \frac{d^2y}{dx^2} = -\frac{r}{x\sqrt{2rx - x^2}}.$$

Substituting these values in the differential equations of the circle, we obtain

Fig. 17



$$x - a + (y - b)\sqrt{\frac{2r - x}{x}} = 0;$$

$$\frac{2r}{x} - \frac{(y - b)r}{x\sqrt{2rx - x^2}} = 0.$$

$$\therefore\ y - b = 2\sqrt{2rx - x^2},\qquad y = b + 2\sqrt{2rx - x^2};$$

$$x - a + 2(2r - x) = 0,\quad x = 4r - a.$$

These values of x and y, substituted in the equation of the cycloid, give, for the equation of its evolute,

$$b = r\ \text{versin}^{-1}\frac{4r - a}{r} - \sqrt{2r(4r - a) - (4r - a)^2}.$$

This is the equation of a cycloid equal to the given curve, as will appear by transferring the origin from C to the point A, whose coördinates are

$$a = 2r, \quad b = r \text{ versin}^{-1} 2.$$

We thus have

$$r \text{ versin}^{-1} 2 + b = r \text{ versin}^{-1} \left\{ 2 - \frac{a}{r} \right\} - \sqrt{2ra - a^2}, \text{ or}$$

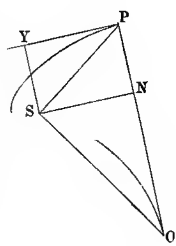
$$b = - r \text{ versin}^{-1} \frac{a}{r} - \sqrt{2ra - r^2},$$

which is the equation of the cycloidal arc AE.

11. The evolute of the logarithmic spiral.

In finding the evolute of a curve whose equation is referred to polar coördinates, it is generally best to transform the equation to r and p as coördinates. For this purpose, let S be the pole, P a point on the given curve, PO the radius of curvature of the point P, PY the tangent at P, SP the radius-vector of P, SY the perpendicular on the tangent PY. Then will O be a point on the evolute, PO the tangent to the evolute at O, SO the radius-vector of O, and SN the perpendicular on PO.

Fig. 18



Let $SP = r$, $SY = p$, $SO = r_{,}$, $SN = p_{,}$, $PO = R$.

Then we shall have

$$SO^2 = SP^2 + PO^2 - 2PO \times PN, \text{ or}$$

$r_{,}^2 = r^2 + R^2 - 2Rp$ (1), since $PN = SY = p$; also

$p_{,} = \sqrt{r^2 - p^2}$ (2),

$p = f(r)$ (3), the equation of the given curve,

and $R = r \dfrac{dr}{dp}$ (4) [Art. 115, Cor. 2].

From the equation of the logarithmic spiral $r = a^{\theta}$, we find $p = nr$, n being a function of the modulus;

$$\therefore \quad R = r\frac{dr}{dp} = \frac{r}{n};$$

$$p_{\prime} = \sqrt{r^2 - p^2} = r\sqrt{1 - n^2}; \quad r_{\prime}^{2} = r^2 + R^2 - 2Rp = \frac{r^2}{n^2} - r^2$$

$$= \frac{r^2}{n^2}(1 - n^2) = \frac{p_{\prime}^{2}}{n^2}.$$

$\therefore$ the equation of the evolute is

$$p_{\prime} = nr_{\prime},$$

which is the equation of a logarithmic spiral similar to the given curve.

CHAPTER XVII.

SINGULAR POINTS OF CURVES.

126. A singular point of a curve is one which possesses some peculiarity which distinguishes it from other points of the curve.

Such points are the points of greatest and least curvature; the points where the tangent is parallel or perpendicular to either of the axes; **multiple** points, or points through which several branches of the curve pass; **points of inflection**, or points where the curvature changes from convexity to concavity; **cusps**, or points where two branches of the curve, which are tangent to each other, terminate; **conjugate** points, or points whose coördinates satisfy the equation of the curve, while the points themselves are entirely detached from the curve; **stop** points, or points at which a branch of the curve suddenly terminates; **salient**

points, or points at which two branches terminate without being tangent. The Calculus affords a very simple means of detecting and determining the positions of such points; and when they are found, the curve may be readily traced through them.

127. Problem.—*To find the points at which a given curve is parallel or perpendicular to the axis of* x.

Let $u = F(x, y) = 0$ be the equation of the curve.

Then $du = \frac{du}{dx}dx + \frac{du}{dy}dy = 0$, and $\frac{dy}{dx} = -\frac{du}{dx} \div \frac{du}{dy}$.

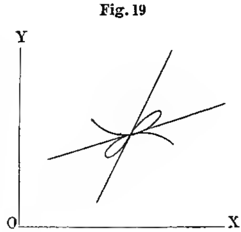
If the curve is parallel to the axis of x at the point x', y', then $\frac{dy'}{dx'} = 0$; and if it is perpendicular to the axis of x at the point x', y', then $\frac{dy'}{dx'} = \infty$. The values of x and y, which satisfy the two equations $\frac{dy}{dx} = 0$, and $\frac{dy}{dx} = \infty$, are the coördinates of the required points.

128. Problem.—*To find the multiple points of a curve.*

Let $u = F(x, y) = 0$ be the equation of the curve.

Then $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$ (a).

Fig. 19



Now, since a multiple point is a point through which several branches of the curve pass, each branch will have a tangent line at that point, and, consequently, $\frac{dy}{dx}$ must have several values.

Hence, since $\frac{du}{dx}$ and $\frac{du}{dy}$ remain fixed, equation (a) can not be satisfied for more than one value of $\frac{dy}{dx}$ unless $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$.

We must therefore have, for a multiple point, the three equations

$$u = F(x, y) = 0 \quad (1); \qquad \frac{du}{dx} = 0 \quad (2); \qquad \frac{du}{dy} = 0 \quad (3);$$

and the coördinates of multiple points are to be found among those real values of x and y which satisfy these three equations.

Again, since $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, the expression for $\frac{dy}{dx}$ will assume the indeterminate form $\frac{0}{0}$, and its values may be found by the rules established for such cases; or they may be found by differentiating equation (1) several times, and placing equal to zero the *first* differential equation so found in which the values of $\frac{dy}{dx}$ are *determinate.*

EXAMPLES.

1\. Find the multiple points of the curve

$$u = x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$

We have

$$\frac{du}{dx} = 4x^3 - 4a^2x = 0, \qquad \therefore\ x = 0,\ x = \pm a;$$

$$\frac{du}{dy} = -6ay^2 - 6a^2y = 0, \quad \therefore\ y = 0,\ y = -a.$$

The values of x and y which satisfy the equation of the curve are

$$x = 0,\ y = -a; \quad x = a,\ y = 0; \quad x = -a,\ y = 0;$$

and these are the coördinates of the only points to be examined.

Now, by differentiating the equation of the curve twice, we have

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2=0\,;$$

and, forming the partial second derivatives from the equation of the curve, and substituting them in this last equation, we obtain

$$12x^2-4a^2-(12ay+6a^2)\left(\frac{dy}{dx}\right)^2=0; \text{ whence}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{6x^2-2a^2}{6ay+3a^2}=\frac{4}{3} \text{ when } x=\pm a,\ y=0,$$

$$=\frac{2}{3} \text{ when } x=0,\ y=-a.$$

$$\therefore \left(\frac{dy}{dx}\right)=\pm\sqrt{\frac{4}{3}} \text{ when } x=+a,\ y=0,$$

$$=\pm\sqrt{\frac{4}{3}} \text{ when } x=-a,\ y=0,$$

$$=\pm\sqrt{\frac{2}{3}} \text{ when } x=0,\ y=-a.$$

The three points are therefore all multiple points, and through each pass *two* branches of the curve.

2. The curve $u=x^4+2ax^2y-ay^3=0$ . . (1).

We have

$$\frac{du}{dx}=4x^3+4axy=0, \quad \therefore\ x=0,\ y=0;$$

$$\frac{du}{dy}=2ax^2-3ay^2=0, \quad \therefore\ x=0,\ y=0;$$

and these are the only values of x and y which satisfy all three equations.

Differentiating the equation of the curve $u = F(x, y) = 0$ three times, we have

$$\frac{d^3u}{dx^3} + 3\frac{d^3u}{dx^2dy}\frac{dy}{dx} + 3\frac{d^3u}{dy^2dx}\left(\frac{dy}{dx}\right)^2 + \frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^3 = 0 \quad . \; . \quad (2);$$

and from the equation of the curve,

$$\frac{d^3u}{dx^3} = 24x; \quad \frac{d^3u}{dx^2dy} = 4a; \quad \frac{d^3u}{dy^2dx} = 0; \quad \frac{d^3u}{dy^3} = -6a.$$

Substituting in (2), and making $x = 0$, we have

$$2\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^3.$$

$$\therefore \; \frac{dy}{dx} = 0, \text{ and } \frac{dy}{dx} = \pm\sqrt{2}.$$

The origin, $x = 0$, $y = 0$, is therefore a *triple* point.

NOTE.—In this example we have differentiated three times, because the values of $\frac{dy}{dx}$ obtained from both the first and second differential equations are indeterminate for the particular values of x and y.

3. The curve $u = x^4 + x^2y^2 - 6ax^2y + a^2y^2 = 0$.

We have

$$\frac{du}{dx} = 4x^3 + 2xy^2 - 12axy = 0, \quad \therefore \; x = 0, \; y = 0;$$

$$\frac{du}{dy} = 2x^2y - 6ax^2 + 2a^2y = 0, \quad \therefore \; x = 0, \; y = 0.$$

Forming the second derivatives, and substituting in the equation

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2=0,$$

we have

$$\frac{dy^2}{dx^2}=0, \quad \therefore \frac{dy}{dx}=\pm\sqrt{0}=\pm 0.$$

The origin is therefore a *double* point, and the two branches at that point have a common tangent, the axis of x.

4. The lemniscata, $u=(x^2+y^2)^2-a^2x^2+a^2y^2=0$.

We find the origin to be a *double* point, and $\dfrac{dy}{dx}=\pm 1$.

5. The folium of Descartes, $u=y^3+x^3-3cxy=0$.

We find

$$\frac{dy}{dx}=0 \text{ or } \infty \text{ for } x=0, \text{ and } y=0.$$

Therefore the origin is a double point, and the two axes are tangent to the curve at that point.

129. Problem.—*To find the conjugate points of a curve.*

It follows from the definition of a conjugate point that if x, y be the coördinates of such a point, the increment Δy, corresponding to an increment Δx, will be *imaginary;* and this being the case, the limit to the ratio of Δy and Δx will usually be imaginary, and, consequently, will admit of at least two values.

We may therefore, in general, determine the coördinates of a conjugate point in the same manner as those of a

multiple point; and if the values of $\frac{dy}{dx}$ prove to be imaginary for any real values of x and y, the corresponding points are certainly conjugate.

But even though Δy should be imaginary, $\frac{dy}{dx}$ may be real, and in such cases we may determine whether a point is conjugate by substituting $x \pm h$ for x in the equation of the curve. If the value of y, corresponding to $x \pm h$, be imaginary, the point is conjugate.

EXAMPLES.

1. The curve $ay^2 - x^3 + bx^2 = 0$.

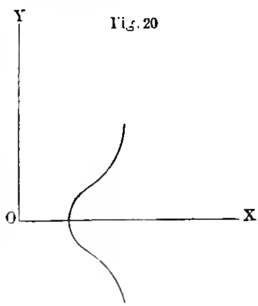
We have

$$\frac{dy}{dx} = \frac{3x^2 - 2bx}{2ay} = \frac{0}{0} \text{ if } x = 0 \text{ and } y = 0;$$

$$= \frac{6x - 2b}{2a\frac{dy}{dx}} = -\frac{b}{a\frac{dy}{dx}}. \text{ Therefore,}$$

$$\left(\frac{dy}{dx}\right)^2 = -\frac{b}{a}, \text{ and } \frac{dy}{dx} = \pm\sqrt{-\frac{b}{a}}.$$

Fig. 20



The origin is therefore a conjugate point.

2. The curve $(c^2y - x^3)^2 = (x - a)^5 (x - b)^6$.

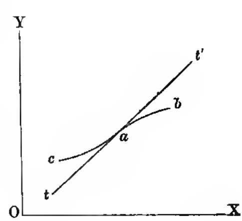
The value of $\frac{dy}{dx}$ in this equation is real for the values of x and y, which evidently satisfy the equation, viz:

$$x = a,\ y = \frac{a^3}{c^2} \text{ and } x = b,\ y = \frac{b^3}{c^2}.$$

If, however, we substitute $b \pm h$ for x in the given equation, the resulting values of y are imaginary. The point $x = b$, $y = \frac{b^3}{c^2}$ is therefore a conjugate point.

130. Problem.—*To find the points of inflection of a curve.*

Fig. 21



The direction of curvature being determined by the sign of $\frac{d^2y}{dx^2}$, it follows that, in passing through a point of inflection, $\frac{d^2y}{dx^2}$ must change its sign, and, therefore, for a point of inflection we must have

$$\frac{d^2y}{dx^2} = 0 \quad (1), \qquad \text{or} \quad \frac{d^2y}{dx^2} = \infty \quad (2).$$

But, as a quantity does not necessarily change its sign in passing through zero or infinity, it follows that the value of $\frac{d^2y}{dx^2}$ may be zero or infinity without the corresponding point being a point of inflection. We must, therefore, after finding the values of x and y which satisfy (1) or (2), substitute $x \pm h$ for x in the value of $\frac{d^2y}{dx^2}$. If these substitutions cause $\frac{d^2y}{dx^2}$ to change sign, the point under consideration is a point of inflection.

Examples.

1. The curve $\qquad a^2y = x^3.$

We have $\frac{dy}{dx} = \frac{3x^2}{a^2}$, $\frac{d^2y}{dx^2} = \frac{6x}{a^2} = 0$ when $x = 0$. $0 + h$ renders $\frac{d^2y}{dx^2}$ positive, and $0 - h$ renders it negative. Therefore the point $x = 0$, $y = 0$, is a point of inflection.

2. The curve $y = 3x + 18x^2 - 2x^3$.

We have

$$\frac{dy}{dx} = 3 + 36x - 6x^2 : \frac{d^2y}{dx^2} = 36 - 12x = 0, \text{ when } x = 3.$$

$$\text{If } x = 3 + h, \ \frac{d^2y}{dx^2} \text{ is negative;}$$

$$\text{and if } x = 3 - h, \ \frac{d^2y}{dx^2} \text{ is positive.}$$

Hence, the point, $x = 3$, $y = 117$, is a point of inflection.

131. Problem.—*To determine the cusps of a curve.*

Since a cusp is a point at which *two* branches of a curve terminate, it follows that it is a multiple point. But since the two branches are tangent to each other, they possess at that point a common tangent line; and, therefore, at that point $\frac{dy}{dx}$ must have *two real* and *equal values.*

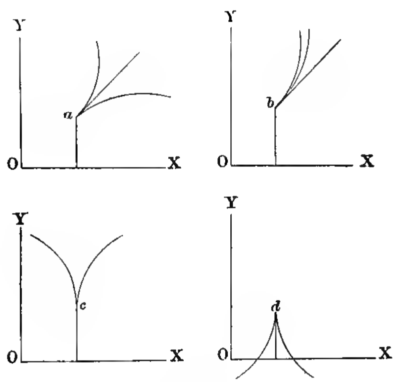


Fig. 22

Again, since the branches terminate at the cusp, the ordinates to the curve will be *real* on one side of the cusp,

and *imaginary* on the other side. Therefore, if x is the abscissa of a cusp point, and $x+h$ renders y real, $x-h$ will render y imaginary.

Again, if the two branches of the curve lie on the same side of their common tangent, their curvatures will be of the same character; and, therefore, the values of $\frac{d^2y}{dx^2}$ for the two branches will have the same sign. Such a cusp is called a *ramphoid* cusp.

If the two branches lie on opposite sides of the tangent, their curvatures will have opposite characters, and the two values of $\frac{d^2y}{dx^2}$ will have contrary signs. Cusps of this character are called *ceratoid* cusps.

Note.—In the particular case in which we find the common tangent to the two branches to be perpendicular to the axis of x, it is generally best to consider y as the independent variable, and find the values of $\frac{d^2x}{dy^2}$.

Examples.

1. $$u = y^2 - 3x^3 = 0.$$

We have

$$\frac{du}{dx} = -9x^2 = 0, \quad \therefore \; x = 0;$$

$$\frac{du}{dy} = 2y = 0, \qquad \therefore \; y = 0.$$

$$\frac{d^2u}{dx^2} = -18x; \quad \frac{d^2u}{dx\,dy} = 0; \quad \frac{d^2u}{dy^2} = 2.$$

$$\therefore \; -18x + 2\left(\frac{dy}{dx}\right)^2 = 0, \text{ and } \frac{dy}{dx} = \pm 0 \text{ when } x = 0.$$

$x = 0 - h$ renders y imaginary, and $\frac{d^2y}{dx^2}$ changes sign with y. Hence, the origin is a cusp, the axis of abscissas is the common tangent, and the two branches lie on opposite sides of this tangent.

2. $$y = b + cx^2 + (x - a)^{\frac{5}{2}}.$$

We have

$$\frac{dy}{dx} = 2cx + \frac{5}{2}(x - a)^{\frac{3}{2}}; \quad \frac{d^2y}{dx^2} = 2c + \frac{15}{4}(x - a)^{\frac{1}{2}}.$$

The values $x = a$, $y = b + ca^2$, satisfy the equation of the curve; and, in consequence of the fractional exponent in the second term of $\frac{dy}{dx}$, this derivative has two equal values, $2ca$, for $x = a$.

If $x > a$, y is real; and if $x < a$, y is imaginary.

If $x = a + h$, $\frac{d^2y}{dx^2} = 2c + \frac{15}{4}h^{\frac{1}{2}}$, which has two values, both positive when h is an infinitesimal. Therefore the point $x = a$, $y = b + ca^2$, is a cusp; the two branches are both convex to the axis of x, and they are situated on the same side of their common tangent.

3. The curve $\quad y^4 - axy^2 + x^4 = 0.$

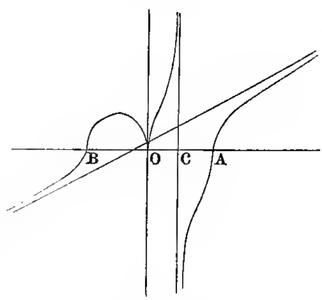
We shall find a ceratoid cusp at the origin.

132. The remaining singular points, such as *stop* points and *salient* points are but seldom met with. According to the definition of a stop point, its test will be similar to that for a cusp, except that $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ will each have but a *single* value. The test for a salient point is the same as that for a cusp, except that $\frac{dy}{dx}$ has two *unequal* values.

133. Tracing of Curves.—If we have given the equation of a curve, the form of the curve may be traced by determining the character and position of all of its singular points, the number and directions of its asymptotes, and as many of its ordinary points as may be convenient. Having located these points, the curve may be drawn through them by hand with a considerable degree of accuracy.

It will frequently occur that, by a simple inspection of the equation of the curve, we can determine many of the properties of the curve itself, and it is advisable to do this whenever it is practicable. We append a few examples.

Fig. 23



1. Trace the curve whose equation is

$$y^3 = \frac{x^4 - a^2 x^2}{2x - a}.$$

1*st.* When $x = 0$, $y = 0$.

$\therefore$ the curve passes through the origin.

2*d.* When $x = \frac{a}{2}$, $y = \pm \infty$; when $x = \pm \infty$, $y = \pm \infty$.

$\therefore$ the curve extends to infinity in *four* directions.

3*d.* When $x = \pm a$, $y = 0$.

$\therefore$ the curve cuts the axis of x at two points.

4*th.* Since $y = \infty$ when $x = \frac{a}{2}$, the line whose equation is $x = \frac{a}{2}$ is an asymptote to the curve.

5th. If we expand the value of y^3, and extract the cube root of the result, we shall have

$$y = \frac{1}{\sqrt[3]{2}}\left\{x + \frac{a}{6}\right\} +$$

terms involving negative powers of x.

$\therefore$ the line whose equation is

$$y = \frac{1}{\sqrt[3]{2}}\left\{x + \frac{a}{6}\right\}$$

is a second asymptote.

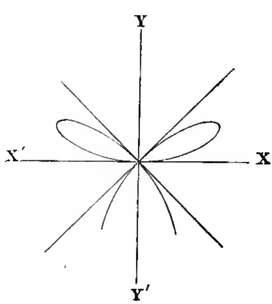
6th. Forming the first derivative, we have

$$\frac{dy}{dx} = \frac{6x^4 - 4ax^3 - 2a^2x^2 + 2a^3x}{3\,(2x-a)^{\frac{4}{3}}\,(x^4 - a^2x^2)^{\frac{2}{3}}} = \infty \text{ when } x = 0,\ \frac{a}{2},\ \pm a;$$

$$= 0 \text{ when } x < 0 \text{ and } > -a.$$

Hence, at the points corresponding to these values of x the tangents are respectively perpendicular and parallel to the axis of x.

Fig. 24



7th. We shall find points of inflection at the points whose abscissas are $x = a$, $x = -a$, $x > 0$ and $< \frac{a}{2}$, and a cusp at the origin.

2. The curve

$$x^4 - ayx^2 + ay^3 = 0.$$

We shall find $\frac{dy}{dx} = 0$ and ± 1, when $x = 0$, $y = 0$.

$\therefore$ the origin is a triple point.

We shall find no asymptotes.

3. The folium of Descartes, $y^3 - 3cxy + x^3 = 0$.

4. The curve $y^2 = \dfrac{a^4}{a^2 + x^2}$.

5. The curve $y = 2a\sqrt{\dfrac{2a - x}{x}}$.

6. The curve $y^2 = \dfrac{x^3 + 1}{x^2 - 1}$.

7. The curve $(y^2 + x^2)^2 = a^2(x^2 - y^2)$.

8. The curve $r = a(1 - \cos\theta)$.

This curve can be traced by assigning different values to θ, and determining the resulting values of r. In the same manner may be traced all curves given by their polar equations.

CHAPTER XVIII.

ENVELOPES OF CURVES.

134. If in the equation of any plane curve $u = F(x, y, a)$, we assign to the arbitrary constant a a series of different values, the resulting equations will be the equations of a series of curves different from each other in form and position, but all belonging to the same class.

If we suppose a to change by infinitesimal amounts, the curves of the series will differ in position by infinitesimal amounts; and, as a general rule, any two adjacent curves of the series will intersect. Let us suppose that the points of intersection of every two adjacent curves are connected by straight lines. There will thus be formed a polygon, the lengths of whose sides will depend upon the actual

value of the increment assigned to a. Now, since this increment is an infinitesimal, it is obvious that the smaller it becomes, the more will the polygon tend toward coincidence with a certain determinate curve; and as the limit to the increment is *zero*, the limit to the polygon is this curve.

This curve is called the **envelope** of the series, and it may be defined to be *the locus of the limiting points of intersection of the consecutive curves of the series.*

135. It is clear that when *all* the consecutive curves of the series intersect each other two and two, the sides of their polygon of intersection will approach toward coincidence with their common tangents as the points of intersection approach each other, and that the limiting curve, or envelope, *will be tangent to all the curves of the series.* But, as the consecutive curves do not necessarily intersect, the envelope is not necessarily tangent to all of the curves.

136. Problem. — *To determine the equation of the envelope of a given series of curves.*

Let $F(x, y, a) = 0$, and $F(x, y, a + h) = 0$, be the equations of two adjacent curves of the series.

The values of x and y which satisfy these two equations are evidently the coördinates of the point of intersection of the two curves, and they will converge toward the values of the coördinates of the limiting point of intersection as h tends toward *zero.*

If, therefore, we combine the equations so as to eliminate a, and then pass to the limit by making h equal to zero, the resulting equation will be the equation of the locus of the limiting points of intersection, or of the envelope.

Now, if we subtract the first equation from the second, we shall have a new equation, which may be employed instead of the second without affecting the result.

We have, therefore,

$$F(x, y, a+h) - F(x, y, a) = 0;$$

or, dividing by h and passing to the limit,

$$\frac{d\{F(x, y, a)\}}{da} = 0.$$

Hence, to find the equation of the envelope it is sufficient to eliminate a between the two equations

$$F(x, y, a) = 0, \text{ and } \frac{d\{F(x, y, a)\}}{da} = 0.$$

EXAMPLES.

1. The envelope of a series of parabolas whose vertices and axes are the same.

We have

$$y^2 = 2px, \text{ or } u = y^2 - 2px = 0.$$

$$\therefore \frac{du}{dp} = -2x = 0.$$

These two equations give us for the equation of the envelope $x = 0$.

The envelope is therefore the axis of y, as might have been inferred.

2. The envelope of a series of circles whose centers are all on the same straight line, the axis of x, and whose radii are proportional to the distances of their centers from the origin.

Let $u = (x - a)^2 + y^2 - r^2 = 0$ be the equation of one of the circles.

Also, let $r = ca$, or $r^2 = c^2a^2$.

Then, $$u = (x - a)^2 + y^2 - c^2a^2 = 0;$$

$$\frac{du}{da} = -2(x - a) - 2ac^2 = 0.$$

$$\therefore \quad a = \frac{x}{1 - c^2},$$

and, by substitution in the equation of the circle, we have, as the equation of the envelope,

$$y = \frac{\pm cx}{1 - c^2}.$$

This is the equation of two straight lines passing through the origin, and making equal angles with the axis of x.

These two lines constitute the envelope of the series of circles, as may be readily seen.

3. The envelope of a series of concentric ellipses, the sum of whose axes is constant, the axes of all the curves of the series being coincident in direction.

The equation of one of the curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and we have the condition $a + b = c$.

Differentiating these equations with respect to a, we have

$$\frac{x^2}{a^3}\,da + \frac{y^2}{b^3}\,db = 0;$$

$$da + db = 0.$$

$$\therefore \quad \frac{db}{da} = -1, \text{ and } \frac{x^2}{a^3} = \frac{y^2}{b^3}.$$

$$\therefore \quad \frac{\left\{\frac{x^2}{a^2}\right\}}{a} = \frac{\left\{\frac{y^2}{b^2}\right\}}{b} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a+b} = \frac{1}{c}:$$

whence,

$$\frac{x^3}{a^3} = \frac{x}{c}, \quad \frac{x^2}{a^2} = \left\{\frac{x}{c}\right\}^{\frac{2}{3}}; \quad \frac{y^3}{b^3} = \frac{y}{c}, \quad \frac{y^2}{b^2} = \left(\frac{y}{c}\right)^{\frac{2}{3}};$$

and
$$\left(\frac{x}{c}\right)^{\frac{2}{3}} + \left(\frac{y}{c}\right)^{\frac{2}{3}} = 1,$$

the equation of the envelope.

4. A given straight line slides between two rectangular axes; find the equation of the curve to which it is always tangent.

Let c be the length of the line, a and b its intercepts on the axes.

Then the equation of the line will be

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and we shall also have the condition

$$a^2 + b^2 = c^2.$$

Differentiating with respect to a, we have

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0; \quad a + b\frac{db}{da} = 0, \quad \frac{db}{da} = -\frac{a}{b}.$$

$$\therefore \quad \frac{x}{a^2} - \frac{ay}{b^3} = 0, \quad b = a\sqrt[3]{\frac{y}{x}}; \quad a^2 + b^2 = c^2 = a^2\left\{\frac{y^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\},$$

$$a = \frac{cx^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}, \quad b = \frac{cy^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}};$$

and by substitution in the equation of the given line, and reduction, we have, as the equation of the envelope,

$$\left(\frac{x}{c}\right)^{\frac{2}{3}} + \left(\frac{y}{c}\right)^{\frac{2}{3}} = 1.$$

5. The envelope of a series of equal circles whose centers all lie on the circumference of a fixed circle.

Let $x_{,}^2 + y_{,}^2 - r_{,}^2 = 0$ (1), be the equation of the fixed circle; and $(x - x_{,})^2 + (y - y_{,})^2 - r^2 = 0$ (2), be the equation of one of the movable circles.

Differentiating with respect to $x_{,}$, we obtain

$$x_{,} + y_{,}\frac{dy_{,}}{dx_{,}} = 0, \text{ or } \frac{dy_{,}}{dx_{,}} = -\frac{x_{,}}{y_{,}} = -\frac{x_{,}}{\sqrt{r_{,}^2 - x_{,}^2}} \quad (3);$$

$$-2(x - x_{,}) - 2(y - y_{,})\frac{dy_{,}}{dx_{,}} = 0 \qquad (4).$$

Combining (1), (2), (3), and (4), so as to eliminate $x_{,}$ and $y_{,}$, we have

$$x^2 + y^2 = (r_{,} \pm r)^2.$$

This is the equation of two concentric circles which constitute the envelope.

6. The envelope of the consecutive normals to an ellipse.

Let x', y' be a point on the ellipse.

Then
$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x') \qquad (1)$$

is the equation of the normal line; and the given point being on the curve, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \qquad (2).$$

Differentiating (1) and (2) with respect to x', and eliminating y' between the resulting equation and (2), we find

$$x'^2 = \frac{a^{\frac{8}{3}}\, x^{\frac{2}{3}}}{(a^2 - b^2)^{\frac{2}{3}}}.$$

Differentiating (1) and (2) with respect to y', and eliminating x', we obtain

$$y'^2 = \frac{b^{\frac{8}{3}}\, y^{\frac{2}{3}}}{(a^2 - b^2)^{\frac{2}{3}}}.$$

Substituting these values of x'^2 and y'^2 in (2), we have, for the equation of the envelope,

$$a^{\frac{2}{3}}\, x^{\frac{2}{3}} + b^{\frac{2}{3}}\, y^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

which we have previously found to be the equation to the *evolute of the ellipse.*

The result established in this example for the ellipse is true for all curves whatever, as might be readily proved.

CHAPTER XIX.

TANGENTS AND NORMALS TO CURVES OF DOUBLE CURVATURE AND SURFACES. DIFFERENTIAL EQUATIONS OF SURFACES.

137. A curve of double curvature is one all of whose points are not in the same plane. It may be determined by the equations of its projections on any two of three coördinate planes, or by the equations of the surfaces of which it is the intersection.

The tangent line to such a curve, at a given point, is the limit to the secant line drawn through that point and another at an infinitesimal distance from it.

The length of the curve, or of any portion of it, is the limit to the length of the broken line or polygon inscribed in it.

Let x, $x+\Delta x$; y, $y+\Delta y$; z, $z+\Delta z$; be the coördinates of two points on the curve, and let Δs be the length of the chord joining them, Δx, etc., being infinitesimals. Then, supposing the coördinate planes to be rectangular, we shall have

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2};$$

and passing to the limit, observing that the limit to the ratio of the chord and arc is unity, we have

$$d\text{ arc} = \sqrt{dx^2 + dy^2 + dz^2} \qquad (1).$$

138. Problem.—*To find the equation of a tangent line to a curve of double curvature.*

The equations of the *secant* line passing through the points whose coördinates are x', $x'+\Delta x'$, y', $y'+\Delta y'$, z', $z'+\Delta z'$, are

$$x - x' = \frac{\Delta x'}{\Delta z'}(z - z'); \quad y - y' = \frac{\Delta y'}{\Delta z'}(z - z');$$

and since the tangent is the limit to the secant, we have, by passing to the limit,

$$x - x' = \frac{dx'}{dz'}(z - z'); \quad y - y' = \frac{dy'}{dz'}(z - z') \qquad (2),$$

the equations of the tangent line.

Corollary 1.—If the curve be given by the equations of two intersecting surfaces,

$$F(x, y, z) = 0, \quad f(x, y, z) = 0,$$

we shall have

$$\frac{dF}{dx'}\frac{dx'}{dz'} + \frac{dF}{dy'}\frac{dy'}{dz'} + \frac{dF}{dz'} = 0,$$

$$\frac{df}{dx'}\frac{dx'}{dz'} + \frac{df}{dy'}\frac{dy'}{dz'} + \frac{df}{dz'} = 0.$$

Finding the values of $\frac{dx'}{dz'}$, $\frac{dy'}{dz'}$, and substituting in (2), we have

$$\left.\begin{aligned}(x - x')\frac{dF}{dx'} + (y - y')\frac{dF}{dy'} + (z - z')\frac{dF}{dz'} = 0\\ (x - x')\frac{df}{dx'} + (y - y')\frac{df}{dy'} + (z - z')\frac{df}{dz'} = 0\end{aligned}\right\} \quad (3);$$

another form for the equations of the tangent.

Corollary 2.—The secant passing through the two points will make, with the three axes, angles whose cosines are, respectively,

$$\frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s}, \frac{\Delta z}{\Delta s}.$$

Hence the tangent will make with the same axes angles whose cosines are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}.$$

Representing these angles by α, β, γ, we have

$$\cos\alpha = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2 + dy^2 + dz^2}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}};$$

$$\cos\beta = \frac{dy}{ds} = \frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}} = \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2}};$$

$$\cos\gamma = \frac{dz}{ds} = \frac{dz}{\sqrt{dx^2 + dy^2 + dz^2}} = \frac{\frac{dz}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2}}.$$

139. A normal plane is a plane perpendicular to the tangent line at the point of tangency.

The equation of a plane passing through a point x', y', z', being of the form

$$A(x - x') + B(y - y') + C(z - z') = 0,$$

$$\text{or } \frac{A}{C}(x - x') + \frac{B}{C}(y - y') + (z - z') = 0;$$

if the plane is perpendicular to the tangent, we must have

$$\frac{A}{C} = \frac{dx'}{dz'}; \quad \frac{B}{C} = \frac{dy'}{dz'}. \quad \text{Hence,}$$

$$\left.\begin{array}{l} (x - x')\dfrac{dx'}{dz'} + (y - y')\dfrac{dy'}{dz'} + (z - z') = 0, \text{ or} \\ (x - x')\,dx' + (y - y')\,dy' + (z - z')\,dz' = 0 \end{array}\right\} \quad (4),$$

is the equation of the normal plane passing through the point x', y', z'.

Every line in the normal plane, which passes through the point of intersection of the plane and curve, is called a *normal line.*

Every plane which passes through the tangent line is called a tangent plane. It is evident that tangent planes will, in general, intersect the curve.

140. Tangents and Normals to Surfaces.—A *tangent plane* to a surface at any given point is the locus of all the tangent lines to the surface which can be drawn through that point. We shall determine the equation of this locus, and at the same time show that it is a plane.

Let $u = F(x, y, z) = 0$, or $z = f(x, y)$, be the equation of a surface.

If a curve be traced through any point x', y', of this surface, it will, in general, be a curve of double curvature; and a tangent line to the curve, at the point x', y', will also be tangent to the surface.

The differential equations of this tangent line will be

$$(x - x') = \frac{dx'}{dz'}(z - z'); \quad (y - y') = \frac{dy'}{dz'}(z - z').$$

Differentiating the second form of the equation of the surface, we have

$$1 = \left(\frac{dz'}{dx'}\right)\frac{dx'}{dz'} + \left(\frac{dz'}{dy'}\right)\frac{dy'}{dz'}$$

in which the ( )s denote the partial derivatives of z, with respect to x' and y'.

If we eliminate $\frac{dx'}{dz'}, \frac{dy'}{dz'}$, from the last three equations, we have, as the equation of the required locus,

$$(z - z') = (x - x')\frac{dz'}{dx'} + (y - y')\frac{dz'}{dy'} \qquad (5);$$

and since this equation is of the first degree between the variables x, y, z, it follows that it is the equation of a plane.

Corollary.—If we take the first form of the equation of the surface $u = F(x, y, z)$, we have

$$\frac{dF}{dx'} + \frac{dF}{dz'}\frac{dz'}{dx'} = 0;$$

$$\frac{dF}{dy'} + \frac{dF}{dz'}\frac{dz'}{dy'} = 0.$$

Finding the values of $\frac{dz'}{dx'}$, $\frac{dz'}{dy'}$, from these equations, and substituting in (5), we obtain

$$(x - x')\frac{dF}{dx'} + (y - y')\frac{dF}{dy'} + (z - z')\frac{dF}{dz'} = 0 \quad (6);$$

another form for the equation of the tangent plane.

Comparing this with equations (3), we learn that *the tangent line to a curve of double curvature*, at a given point of that curve, *is the intersection of the two planes which are tangent* at that point *to the surfaces of which the curve is the intersection.*

141. A normal line to a curved surface is a line perpendicular to a tangent plane at the point of tangency.

Its equations are, therefore,

$$(x - x') = -\frac{dz'}{dx'}(z - z');$$

$$(y - y') = -\frac{dz'}{dy'}(z - z') \quad . \; . \; . \quad (7);$$

$$\text{or } (x - x')\frac{dF}{dz'} = (z - z')\frac{dF}{dx'};$$

$$(y - y')\frac{dF}{dz'} = (z - z')\frac{dF}{dy'} \quad . \; . \; . \quad (8).$$

Corollary.—If we designate by θ', θ'', θ''', the angles which a normal line makes with the axes, or those which the tangent plane makes with the three coördinate planes, we shall have [Art. 138, Cor.]

$$\cos\theta' = \frac{\dfrac{dz'}{dx'}}{\sqrt{1+\left(\dfrac{dz'}{dx'}\right)^2+\left(\dfrac{dz'}{dy'}\right)^2}},$$

$$\cos\theta'' = \frac{\dfrac{dz'}{dy'}}{\sqrt{1+\left(\dfrac{dz'}{dx'}\right)^2+\left(\dfrac{dz'}{dy'}\right)^2}},$$

$$\cos\theta''' = \frac{1}{\sqrt{1+\left(\dfrac{dz'}{dx'}\right)^2+\left(\dfrac{dz'}{dy'}\right)^2}}.$$

142. A normal plane is a plane perpendicular to a tangent line at the point of tangency.

The equations of the tangent line being

$$(x-x') = \frac{dx'}{dz'}(z-z'); \quad (y-y') = \frac{dy'}{dz'}(z-z'),$$

it follows, at once, that the equation of the normal plane will be

$$(x-x')\frac{dx'}{dz'} + (y-y')\frac{dy'}{dz'} + (z-z') = 0 \qquad (9).$$

143. Differential Equations of Surfaces.—If we resolve the two equations of any curved line with reference to any one of the constants which enter into them, we shall have two new equations of the form $u = c$, $u_{,} = c_{,}$, in which u and $u_{,}$ are functions of x, y, z, and c and $c_{,}$ are constants.

If, in these last equations, we attribute to c and $c_{,}$ a series of *arbitrary* values in succession, the line represented by these equations will change its position, and perhaps its form, without describing any determinate surface. But if we *impose some fixed relation* upon c and $c_{,}$, as $c_{,} = \phi(c)$, the two equations will represent a line whose form and position will be determinate for each particular value of c; and if we eliminate c between these equations, the resulting equation,

$$u_{,} = \phi(u),$$

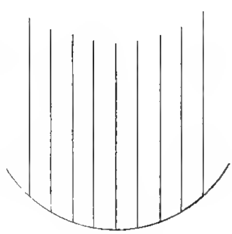
will obviously be the equation of the locus of the entire series of lines, which will necessarily be a surface generated by the motion of the line $u = c$, $u_{,} = c_{,}$, according to the laws prescribed by the relation $c_{,} = \phi(c)$.

The function ϕ may be eliminated by the methods already established for the elimination of functions, and the resulting equation will usually be a differential equation.

Applications.—1*st. To determine the general differential equation of all cylindrical surfaces.*

A cylindrical surface is one which may be generated by the motion of a right line whose consecutive positions are parallel to each other, and which rests constantly upon a given curve.

Fig. 25



Let $x = tz + a$, $y = sz + b$, be the equations of the moving straight line. Then $u = x - tz = a$, $u_{,} = y - sz = b$.

$\therefore$ the equation of the surface described by the line is

$$y - sz = \phi(x - tz),$$

in which s and t are constant by the conditions of the problem.

To eliminate ϕ, differentiate with respect to x and y.

We thus obtain

$$-s\frac{dz}{dx}=\frac{d\phi}{d(x-tz)}\cdot\frac{d(x-tz)}{dx},$$

$$1-s\frac{dz}{dy}=\frac{d\phi}{d(x-tz)}\cdot\frac{d(x-tz)}{dy};$$

whence, by division and reduction,

$$t\frac{dz}{dx}+s\frac{dz}{dy}=1 \qquad (10),$$

the required equation.

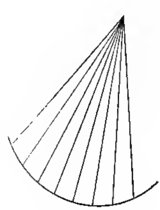
In applying this equation the values of $\frac{dz}{dy}$, $\frac{dz}{dx}$, must be taken from the equation of the fixed curve upon which the straight line rests.

2d. *The general differential equation of all conical surfaces.*

A conical surface is one which may be generated by the motion of a right line which passes through a fixed point and rests upon a given curve.

Let a, b, c, be the coördinates of the fixed point. Then

Fig. 26



$$x-a=t(z-c); \quad y-b=s(z-c)$$

will be the equations of the moving line, in which t and s are variable.

We shall have

$$u=\frac{x-a}{z-c}=t; \quad u_{,}=\frac{y-b}{z-c}=s.$$

$$\therefore \frac{y-b}{z-c}=\phi\left\{\frac{x-a}{z-c}\right\}.$$

Differentiating with respect to x and y, we have

$$-\frac{y-b}{(z-c)^2}\frac{dz}{dx} = \frac{d\phi}{d\left(\frac{x-a}{z-c}\right)} \cdot \frac{d\left(\frac{x-a}{z-c}\right)}{dx}$$

$$= \frac{d\phi}{d\left(\frac{x-a}{z-c}\right)}\left\{\frac{1}{z-c} - \frac{x-a}{(z-c)^2}\frac{dz}{dx}\right\};$$

$$\frac{1}{z-c} - \frac{y-b}{(z-c)^2}\frac{dz}{dy} = \frac{d\phi}{d\left(\frac{x-a}{z-c}\right)} \cdot \frac{d\left(\frac{x-a}{z-c}\right)}{dy}$$

$$= \frac{d\phi}{d\left(\frac{x-a}{z-c}\right)}\left\{-\frac{x-a}{(z-c)^2}\frac{dz}{dy}\right\};$$

whence, by division and reduction,

$$z-c = (x-a)\frac{dz}{dx} + (y-b)\frac{dz}{dy} \qquad (11),$$

the required equation, in which the values of $\frac{dz}{dx}$, $\frac{dz}{dy}$, are to be taken from the equations of the given curve.

3d. The general differential equation of all surfaces of revolution.

The characteristic property of this class of surfaces is that every section made by a plane perpendicular to the axis is a circle whose center is in the axis; and we may suppose the surface to be generated by the motion of a variable circle whose consecutive positions are parallel to each other, and whose center remains constantly upon the axis.

Now, since every section of a sphere made by a plane is a circle, we may take the equations of a sphere and a plane as the two equations of the circle which generates the given surface.

Let $x = tz + a$, $y = sz + b$, be the equations of the axis; then the equation of a plane perpendicular to that line will be

$$z + tx + sy = c.$$

If we take the center of the sphere at the point where the axis of the surface cuts the plane of XY, its equation will be

$$(x - a)^2 + (y - b)^2 + z^2 = r^2,$$

and we shall have, for the equation of the surface,

$$(x - a)^2 + (y - b)^2 + z^2 = \phi(z + tx + sy).$$

In order to eliminate the unknown function ϕ, we might proceed as in the last two cases; but, for the sake of simplicity as well as of variety, we shall adopt another method.

It is obvious, from the nature of a surface of revolution, that the normal line to any point of the surface intersects the axis.

The equations of the normal to a curved surface are

$$x - x' + \frac{dz'}{dx'}(z - z') = 0; \quad y - y' + \frac{dz'}{dy'}(z - z') = 0;$$

and in order that this line may intersect the axis, whose equations are

$$x = tz + a, \quad y = sz + b,$$

we must have the relation

$$\frac{x' - a + z'\dfrac{dz'}{dx'}}{\dfrac{dz'}{dx'} + t} = \frac{y' - b + z'\dfrac{dz'}{dy'}}{\dfrac{dz'}{dy'} + s},$$

each member of which equation is the value of z for the point of intersection.

Reducing this equation, and omitting the accents, we have, for the required equation of all surfaces of revolution,

$$(x - a - tz)\frac{dz}{dy} - (y - b - sz)\frac{dz}{dx}$$
$$+ (x - a)s - (y - b)t = 0 \quad (12),$$

in which the values of $\frac{dz}{dy}$, $\frac{dz}{dx}$, must be taken from the equation of the curve which generates the surface.

144. If the algebraic or finite equation of a surface be given, its differential may be obtained under a more symmetrical form than those given in the preceding article.

Let $u = F(x, y, z) = 0$ be the equation of the surface.

Then $\frac{dz}{dx} = -\frac{du}{dx} \div \frac{du}{dz}$; $\frac{dz}{dy} = -\frac{du}{dy} \div \frac{du}{dz}$.

Substituting these values in the preceding equations, we have, for the equation of *cylindrical surfaces*,

$$t\frac{du}{dx} + s\frac{du}{dy} + \frac{du}{dz} = 0 \quad (13);$$

for the equation of *conical surfaces*,

$$(x - a)\frac{du}{dx} + (y - b)\frac{du}{dy} + (z - c)\frac{du}{dz} = 0 \quad (14);$$

for the equation of *surfaces of revolution*,

$$(x - a - tz)\frac{du}{dy} - (y - b - sz)\frac{du}{dx}$$
$$- \{(x - a)s - (y - b)t\}\frac{du}{dz} = 0 \quad (15).$$

145. Problem.—*To find the equation of the cylinder or cone which incloses a given curved surface.*

If we form the derivatives $\frac{dz}{dx}$, $\frac{dz}{dy}$, or $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, from the equation of the given surface, and substitute them in the differential equation of the cylinder or cone, the resulting equation will evidently be that of a surface which contains the curve of contact of the given surface with the cylinder or cone: and if this equation be combined with that of the given surface, the resulting equation will be the equation of the curve of contact itself. Then, having this equation and the axis of the cylinder or the vertex of the cone, as the case may be, the equation of the cylindrical or conical surface may be found as in the last articles.

146. Envelopes of Surfaces.—If in the equation of a surface, $F(x, y, z, a) = 0$, the constant a be made to vary, we shall have the equations of a series of surfaces, all belonging to the same class, but differing in form and position. Any two adjacent surfaces will usually intersect; and a *polyhedron* may be constructed through the points of intersection. As the increment assigned to a tends toward *zero*, this polyhedron will tend toward a certain determinate surface, which is called the **envelope** of the series. Precisely as was done in the case of envelopes of plane curves may we show that the equation of the envelope of a series of curved surfaces may be found by eliminating a between the equations

$$F(x, y, z, a) = 0, \text{ and } \frac{d\{F(x, y, z, a)\}}{da} = 0.$$

147. EXAMPLES.

1. The equations of the tangent line and normal plane to the helix, or screw.

The equations of this curve are

$$y = a \sin \frac{z}{ma}, \quad x = a \cos \frac{z}{ma},$$

in which a denotes the radius of the base of the cylinder on which the curve is formed, and m is the tangent of the inclination of the curve to the plane of the base.

We have, by differentiation,

$$\frac{dx}{dz} = -\frac{1}{m} \sin \frac{z}{ma}; \quad \frac{dy}{dz} = \frac{1}{m} \cos \frac{z}{ma}.$$

Hence, by substitution in the differential equations of the tangent and normal to a curve of double curvature,

$$x - x' = -\frac{1}{m} \sin \frac{z'}{ma} \{z - z'\},$$

$$y - y' = \frac{1}{m} \cos \frac{z'}{ma} \{z - z'\},$$

are the equations of the tangent line;

$$m(z - z') = (x - x') \sin \frac{z'}{ma} - (y - y') \cos \frac{z'}{ma}$$

is the equation of the normal plane.

2. The equation of a tangent plane to the ellipsoid.

We have

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0;$$

$$\frac{du}{dx} = \frac{2x}{a^2}; \quad \frac{du}{dy} = \frac{2y}{b^2}; \quad \frac{du}{dz} = \frac{2z}{c^2}.$$

Substituting these expressions in the general equation of a tangent plane, we have

$$\frac{2x'}{a^2}(x-x')+\frac{2y'}{b^2}(y-y')+\frac{2z'}{c^2}(z-z')=0, \text{ or}$$

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

3. Find the equations of the tangent line and normal plane to the curved line formed by the intersection of an ellipsoid and sphere, referred to the same center and axes.

The equations of the two surfaces are

$$F(x, y, z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \text{ the ellipsoid;}$$

$$f(x, y, z)=x^2+y^2+z^2-r^2=0, \text{ the sphere.}$$

We have, from these two equations,

$$\frac{dx}{dz}=\frac{a^2}{c^2}\left(\frac{b^2-c^2}{a^2-b^2}\right)\frac{z}{x}; \quad \frac{dy}{dz}=\frac{b^2}{c^2}\left(\frac{c^2-a^2}{a^2-b^2}\right)\frac{z}{y}.$$

Substituting these values in the equations of the tangent line and normal plane to a curve of double curvature, we have, for the equations of the tangent line,

$$\frac{x}{a^2}\left(\frac{x-x'}{b^2-c^2}\right)=\frac{y}{b^2}\left(\frac{y-y'}{c^2-a^2}\right)=\frac{z}{c^2}\left(\frac{z-z'}{a^2-b^2}\right);$$

and for the equation of the normal plane,

$$a^2(b^2-c^2)\frac{x-x'}{x}+b^2(c^2-a^2)\frac{y-y'}{y}$$
$$+c^2(a^2-b^2)\frac{z-z'}{z}=0.$$

4. The equations of the tangent line to the curve formed by the intersection of a sphere with a right cylinder, the radius of the base of which is one-half that of the sphere, and whose surface passes through the center of the sphere.

Taking r as the radius of the base of the cylinder, the center of the sphere as the origin, the axis of z parallel to the axis of the cylinder, and the axis of x, the line passing through the center of the cylinder, we have, for the equations of the curve,

$$x^2 + y^2 + z^2 = 4r^2; \quad y^2 + x^2 = 2rx.$$

We shall find, for the equations of the tangent,

$$y(y - y') = (r - x)(x - x'); \quad z(z - z') = r(x - x').$$

5. Find the equations of the curve of contact of a sphere and cone whose vertex is on the axis of y, and at a distance $2r$ from the origin.

The equation of the sphere is

$$u = x^2 + y^2 + z^2 - r^2 = 0.$$

$$\therefore \quad \frac{du}{dx} = 2x; \quad \frac{du}{dy} = 2y; \quad \frac{du}{dz} = 2z.$$

The coördinates of the vertex of the cone are

$$a = 0; \quad b = 2r; \quad c = 0.$$

Substituting these values in the equation of all conical surfaces, we obtain,

$$x.\, 2x + (y - 2r)\, 2y + z.\, 2z = 0, \quad \text{or } x^2 + y^2 + z^2 - 2ry = 0.$$

This is the equation of a surface which contains the curve of contact; and combining it with the equation of the given sphere, we obtain, for the equations of that curve,

$$y = \frac{r}{2}, \text{ and } x^2 + z^2 = \frac{3r^2}{4}.$$

These are the equations of a circle perpendicular to the axis of y, and at a distance $\frac{r}{2}$ from the origin.

6. The envelope of a series of ellipsoids, the sum of whose axes is constant.

The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the given condition is $a + b + c =$ constant $= k$.

Differentiating with respect to a and b, we have

$$\frac{x^2}{a^3} + \frac{z^2}{c^3}\frac{dc}{da} = 0; \quad \frac{y^2}{b^3} + \frac{z^2}{c^3}\frac{dc}{db} = 0;$$

$$1 + \frac{dc}{da} = 0; \qquad 1 + \frac{dc}{db} = 0.$$

$$\therefore \frac{x^2}{a^3} = \frac{y^2}{b^3} = \frac{z^2}{c^3}, \text{ or } \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{a + b + c} = \frac{1}{k} = \frac{x^2}{a^3}.$$

$$\therefore \frac{x^3}{a^3} = \frac{x}{k}; \ \frac{y^3}{b^3} = \frac{y}{k}; \ \frac{z^3}{c^3} = \frac{z}{k};$$

and, by substitution in the equation of the ellipsoid, we obtain for that of the envelope,

$$\left(\frac{x}{k}\right)^{\frac{2}{3}} + \left(\frac{y}{k}\right)^{\frac{2}{3}} + \left(\frac{z}{k}\right)^{\frac{2}{3}} = 1.$$

7. The envelope of a series of planes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

in which $abc =$ constant $= m^3$.

$$\text{Ans. } xyz = \frac{m^3}{27}.$$

CHAPTER XX.

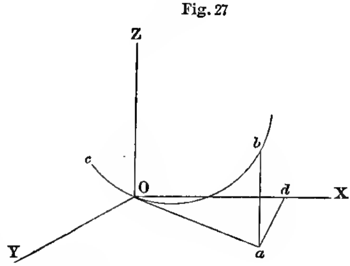
THE CURVATURE OF SURFACES.

148. The curvature of a surface at any point is determined by finding that of the various plane sections passing through that point. Among all these sections, those which are made by normal planes, and which are therefore called *normal sections*, are especially important.

Problem.—*To determine the radius of curvature of a given point of a normal section of a curved surface.*

Let $z = f(x, y)$ be the equation of the surface; and, for the sake of simplicity, let the plane of XY be taken as the tangent plane. The normal plane will then contain the axis of Z.

Fig. 27



Let ZOa be the normal plane, and designate by m the tangent of the angle which this plane makes with the plane of XZ. A circle, situated in the normal plane, and tangent at the origin to the normal section bOc, will have, for its equations,

$$y = mx, \quad x^2 + y^2 + z^2 - 2Rz = 0;$$

whence $$x^2(1 + m^2) + z^2 - 2Rz = 0 \qquad (a).$$

If this circle be the osculatory circle of the section, the values of $\frac{d^2z}{dx^2}$ must be the same for the circle and the section.

Now, equation (a), differentiated twice, gives

$$1 + m^2 + (z - R)\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2.$$

But at the origin we have $z = 0$, $\dfrac{dz}{dx} = 0$; whence

$$1 + m^2 - R\frac{d^2z}{dx^2} = 0. \quad \therefore \quad R = \frac{1 + m^2}{\dfrac{d^2z}{dx^2}} \qquad (1).$$

Differentiating the equation of the surface, and observing that $\dfrac{dy}{dx} = m$, we have,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} + \frac{dz}{dx} = m\frac{dz}{dy} + \frac{dz}{dx};$$

$$\frac{d^2z}{dx^2} = \frac{d^2z}{dx^2} + 2\frac{d^2z}{dx\,dy}m + \frac{d^2z}{dy^2}m^2,$$

in which the values of x, y, z must be placed equal to *zero*, after differentiation, in order to refer the derivatives to the origin.

Designating $\dfrac{d^2z}{dx^2}$ by r, $\dfrac{d^2z}{dx\,dy}$ by s, $\dfrac{d^2z}{dy^2}$ by t, and substituting in (1), we have, for the radius of curvature,

$$R = \frac{1 + m^2}{r + 2sm + tm^2} \qquad (2).$$

149. The greatest and least values of the radii of curvature at any point are called the **principal radii** of that point; and the corresponding sections are called the **principal sections.**

The positions of these sections may be ascertained by determining the values of m which will render R a maximum or minimum.

For this purpose, differentiating (2) with respect to m, and placing the result equal to zero, we have

$$\frac{dR}{dm} = \frac{2sm^2 + 2m(r - t) - 2s}{(r + 2sm + tm^2)^2} = 0.$$

$$\therefore \quad sm^2 + m(r - t) - s = 0 \qquad (3),$$

and by solving this equation,

$$m' = \frac{(t - r) + \sqrt{4s^2 + (t - r)^2}}{2s},$$

$$m'' = \frac{(t - r) - \sqrt{4s^2 + (t - r)^2}}{2s}.$$

These are the two values of m which determine the positions of the principal sections; and since their product is equal to -1, whence

$$m' m'' + 1 = 0,$$

it follows that *the planes of the two principal sections of every point of a curved surface are at right angles to each other.*

150. If we take the planes of these sections as the coördinate planes of XZ and YZ, the expression for R may be much simplified.

For, since in that case the two values of m become 0 or tang 0°, and ∞ or tang 90°, we must have $s = 0$, and therefore the value of R becomes

$$R = \frac{1 + m^2}{r + tm^2} \quad . \quad . \quad . \quad . \quad (4).$$

If we designate by ρ and ρ' the maximum and minimum radii, we shall have

$$\rho = \frac{1}{r} \text{ and } \rho' = \frac{1}{t}.$$

For, putting (4) under the form

$$R = \frac{\frac{1}{m^2} + 1}{\frac{r}{m^2} + t}, \text{ we have for } m = \infty$$

$$R = \frac{1}{t}; \text{ and from (4), when } m = 0, \ R = \frac{1}{r}.$$

Designating by ϕ the angle whose tangent is m, substituting the values of ρ and ρ' in (4), and reducing, we have

$$\frac{1}{R} = \frac{1}{\rho} \cos^2 \phi + \frac{1}{\rho'} \sin^2 \phi.$$

Now, designating by v the curvature of any section, and by v', v'', the curvatures of the principal sections, we have

$$v = \frac{1}{R}, \quad v' = \frac{1}{\rho}, \quad v'' = \frac{1}{\rho'};$$

and, therefore, $v = v' \cos^2 \phi + v'' \sin^2 \phi$ . . . . (5),

a relation between the curvatures of the principal sections of a point and that of any other normal section through the same point.

151. We deduce from (5) the following important consequences:

1*st.* If we pass through the normal two planes which are equally inclined to that of one of the principal sections, the curvatures of the two plane sections will be equal to each other.

For the angles which they make with the plane of the principal section, being designated by ϕ' and $-\phi'$, we shall have, by substituting these two values for ϕ in (5), equal values for v.

2*d*. The sum of the curvatures of any two normal sections which are equally inclined to the principal sections, is constant and equal to the sum of the curvatures of the principal sections.

For, designating the curvatures by v_1, v_2, we have

$$v_1 = v' \cos^2 \phi + v'' \sin^2 \phi,$$

$$v_2 = v' \sin^2 \phi + v'' \cos^2 \phi,$$

$$\therefore\ v_1 + v_2 = v' + v'' = \text{a constant}.$$

3*d*. If $\phi = 45°$ we have $\cos \phi = \sin \phi = \frac{1}{2}\sqrt{2}$.

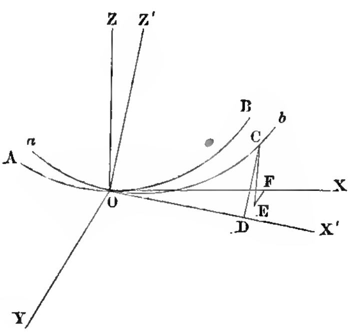
Hence, in this case,

$$v = \frac{v' + v''}{2}.$$

Therefore, the curvature of each of the sections whose planes bisect the angles between the planes of the principal sections is the arithmetical mean of the curvatures of those sections; and, therefore, any two planes which are equally inclined to one of these *medial* planes, give sections the sum of whose curvatures is $v' + v''$, since these planes make equal angles with those of the principal sections.

152. Problem.—*To find the radius of curvature of an oblique section.*

Fig. 28



Let XZ be the normal plane, AOB the normal section, aOb the oblique section made by the plane $Z'Ox'$.

Let $OD = x'$; $DC = z'$; $\alpha =$ angle between XZ and $X'Z'$; $CE = z$; $FE = y$; $OF = x$.

Then, at the point O, we shall have $\frac{d^2z}{dy^2} = 0$, $\frac{d^2z}{dx\,dy} = 0$, and therefore the value of R may be written

$$R = \frac{\left(\frac{ds}{dx}\right)^2}{\frac{d^2z}{dx^2}}, \text{ since } 1 + m^2 = 1 + \frac{dy^2}{dx^2} = \left(\frac{ds}{dx}\right)^2.$$

If we designate the radius of curvature of the oblique section by R', and its arc by s', we shall have

$$R' = \frac{\left(\frac{ds'}{dx'}\right)^2}{\frac{d^2z'}{dx'^2}}.$$

But $\quad z = z' \cos \alpha$, and $\frac{d^2z}{dx^2} = \frac{d^2z'}{dx^2} \cos \alpha.$

Also at the point O,

$$\frac{ds'}{dx'} = \frac{dx'}{dx} = \frac{ds}{dx}.$$

$$\therefore \quad R' = R \cos \alpha.$$

Therefore the *radius of an oblique section* = radius of normal section $\times$ cosine of their included angle = *projection of the radius of the normal section upon the plane of the oblique section.*

THE INTEGRAL CALCULUS.

CHAPTER I.

FIRST PRINCIPLES AND DEFINITIONS.

1. The fundamental conception upon which the **Integral Calculus** is based, is that any magnitude may be considered as the limit to the sum of an infinite series of infinitesimals; and the grand problem of which this Calculus affords the solution may be thus stated: "Given an infinite series of infinitesimals, to find the limit to its sum."

2. Let $y = F(x)$ be any function of x, continuous from y_0 to Y, corresponding to the values x_0 and X of x.

Let the interval $X - x_0$ be divided into a number of parts, each equal to Δx, so that we shall have

$$X = x_0 + \Delta x + \Delta x + \Delta x + \dots + \Delta x.$$

Then, for every increment assumed by x, y will acquire a corresponding increment, which may be represented by Δy, and we shall have

$$Y = y_0 + \Delta y + \Delta y + \Delta y + \dots + \Delta y,$$

in which the different values of Δy will, in general, be unequal to each other.

Now it has been shown in the Differential Calculus [Art. 31], that

$$\Delta y = F'(x)\Delta x + \alpha\Delta x,$$

in which $\alpha\Delta x$ is an infinitesimal of the second order.

Hence, $Y = y_0 + \{F'(x)\Delta x + \alpha\Delta x\} + \{F'(x)\Delta x + \alpha\Delta x\}$
$$+ \{F'(x)\Delta x + \alpha\Delta x\} + \text{etc.};$$

or, $F(X) = F(x_0) + \{F'(x)\Delta x + \alpha\Delta x\}$
$$+ \{F'(x)\Delta x + \alpha\Delta x\} + \text{etc.} \quad (1).$$

Observing that $\lim F(X) = F(X)$, $\lim F(x_0) = F(x_0)$, and that in any series, the limit to whose sum is to be taken, we can replace $F'(x)\Delta x + \alpha\Delta x$ by $F'(x)dx$, we shall have, by taking the limit of (1),

$$F(X) = F(x_0) + \lim \Sigma F'(x)dx \qquad (a);$$

designating by Σ the *sum* of the terms $F'(x)\Delta x + \alpha\Delta x$, etc.

This equation may be abbreviated into

$$F(X) = F(x_0) + \int_{x_0}^{X} F'(x)dx,$$

whence
$$F(X) - F(x_0) = \int_{x_0}^{X} F'(x)dx \qquad (2),$$

in which the second member denotes *the limit to the sum of the series* $F'(x)dx + F'(x)dx +$ etc., and is read the **definite integral** of $F'(x)dx$ between the limits x_0 and X; and in finding the successive terms of the series, $F'(x)$ is to be taken for every possible value of x, from x_0 to X inclusive.

Now $F'(x)dx$ is the differential of $F(x)$, and it follows from the form of equation (2) that *the definite integral of* $\mathrm{F'(x)dx}$, *between the limits* $\mathrm{x_0}$ *and* X, *is to be found by determining the function* $\mathrm{F(x)}$ *whose differential is* $\mathrm{F'(x)dx}$, *and taking the difference between its two values* $\mathrm{F(x_0)}$ *and* $\mathrm{F(X)}$.

3. If, instead of determining the definite integral, we find simply the quantity $F(x)$, of which $F'(x)dx$ is the differential, we shall have the **general,** or indefinite, **inte-**

gral of $F'(x)dx$, which includes the definite integral as a particular case. This integral is denoted thus:

$$\int F'(x)dx = F(x),$$

and the first member is read, *the integral of* $F'(x)dx$.

When an indefinite integral has been thus found, another more general will be determined by adding to the former an arbitrary constant C, since $F'(x)dx$ is the differential not only of $F(x)$ but also of $F(x) + C$.

We thus have, for the most general integral of $F'(x)dx$,

$$\int F'(x)dx = F(x) + C \qquad (3).$$

NOTE. — In the foregoing expressions the symbol $\int$ is merely an elongated S, and it is used only to represent the *limit* to a sum. It will be seen that it bears exactly the same relation to the symbol Σ that d in the Differential Calculus bears to Δ.

4. Having found the expression for the general or indefinite integral, the value of C may be found, provided we can ascertain the value of the integral for any particular value of x.

By the very nature of an integral, the value x_0 of x reduces $\int F'(x)dx$ to *zero*, since the integral may be supposed to originate at that particular value of x.

Hence, we have

$$0 = \int F'(x_0)dx = F(x_0) + C;$$

$$\therefore \quad C = -F(x_0),$$

and by substitution in (3),

$$\int F'(x)dx = F(x) - F(x_0) \qquad (4).$$

A comparison of (2) and (4) teaches us that *the general integral is equivalent to the definite integral taken between the limits* x_0 *and* x, *the latter of which is variable;* and the definite integral between x_0 and X may be found by substituting X for x in the value of the indefinite integral given by equation (4).

5. If in equation (4) we substitute X_1 and X_2 for x, we shall have

$$\int F'(X_2)dx = F(X_2) - F(x_0),$$

$$\int F'(X_1)dx = F(X_1) - F(x_0);$$

and by subtraction,

$$\int F'(X_2)dx - \int F'(X_1)dx = F(X_2) - F(X_1).$$

Now $F(X_2) - F(X_1)$ being the difference between the two definite integrals taken between the limits x_0, X_1, and x_0, X_2, is evidently equal to the definite integral taken between the limits X_1 and X_2; and, in accordance with the adopted notation for definite integrals, we may write

$$\int_{X_1}^{X_2} F'(x)dx = F(X_2) - F(X_1) \qquad (5).$$

We also have, in accordance with the same notation,

$$\int F'(X_2)dx = \int_{x_0}^{X_2} F'(x)dx, \text{ and}$$

$$\int F'(X_1)dx = \int_{x_0}^{X_1} F'(x)dx.$$

$$\text{Hence, } \int_{X_1}^{X_2} F'(x)dx = \int_{x_0}^{X_2} F'(x)dx - \int_{x_0}^{X_1} F'(x)dx \quad (6).$$

We may, therefore, find the definite integral between two given limits X_1 *and* X_2 *by obtaining its values between* x_0, X_1 *and* x_0, X_2, *and subtracting the results.* Or, since the constant C will disappear by subtraction, *we may substitute* X_1 *and* X_2 *for* x *in the value of the general integral, and take the difference between the results.*

INTEGRATION OF FUNCTIONS OF ONE VARIABLE.

CHAPTER II.

DIRECT INTEGRATION, IN GENERAL.

6. The process of finding the indefinite integral, being the same as that of finding the function of which the given function, $F'(x)dx$, is the differential, is obviously the reverse of differentiation; and the general problem to be solved may be restated thus: "Given any differential function, to determine the function of which it is the differential."

The process by which this operation is effected is called **integration.**

7. Every proposition in the Differential Calculus will evidently have its correlative proposition in the Integral Calculus; and it is by reversing the operations of the former that the primary methods and rules of the latter are established.

We shall now present these methods in the shape of *formulas,* which are to be remembered.

Formulas for Direct or Simple Integration.

1. Since $d(ax) = adx$;

$$\therefore \int adx = \int d(ax) = ax + C = a\int dx.$$

Hence, *a constant factor can be moved from one side to the other of the integral sign.*

2. Since $du + dv + dz = d(u + v + z)$;

$$\therefore \int \{du + dv + dz\} = \int d(u + v + z) = u + v + z + C,$$

$$= \int du + \int dv + \int dz.$$

Hence, *the integral of the sum of any number of differentials is equal to the sum of their integrals.*

3. Since $d(uv) = udv + vdu$;

$$\therefore \int \{udv + vdu\} = \int d(uv) = uv + C.$$

4. Since $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$;

$$\therefore \int \frac{vdu - udv}{v^2} = \int d\left(\frac{u}{v}\right) = \frac{u}{v} + C.$$

5. Since $d(x^{m+1}) = (m + 1)x^m dx$, whence

$$x^m dx = \frac{d(x^{m+1})}{m + 1};$$

$$\therefore \int x^m dx = \frac{1}{m+1} x^{m+1} + C.$$

From this formula we obtain the following useful

Rule.—*To integrate a function which is equal to the product of a power of the variable by the differential of the variable, increase the exponent of the variable by unity, and divide the result by the exponent thus increased.*

NOTE.—It will be observed that the rule fails when $m = -1$.

6. Since $d \log x = \frac{dx}{x}$;

$$\therefore \int \frac{dx}{x} = \log x + C.$$

From this formula we derive the following

Rule.—*The integral of a fractional function whose numerator is the differential of the denominator, is equal to the logarithm of the denominator.*

7. Since $d(a^x) = a^x \log a \, dx$;

$$\therefore \int a^x \log a \, dx = a^x + C, \text{ and}$$

$$\int a^x \, dx = \frac{a^x}{\log a} + C.$$

If $a = e$, the Naperian base, we have $\log a = \log e = 1$;

$$\therefore \int e^x dx = e^x + C.$$

8. Since $d \sin x = \cos x \, dx$;

$$\therefore \int \cos x \, dx = \sin x + C.$$

9. Since $d \cos x = - \sin x \, dx$;

$$\therefore -\int \sin x \, dx = \cos x + C.$$

10. Since d tang $x = \sec^2 x\,dx$;

$$\therefore \int \sec^2 x\,dx = \text{tang } x + C.$$

11. Since $d \cot x = - \text{cosec}^2 x\,dx$;

$$\therefore -\int \text{cosec}^2 x\,dx = \cot x + C.$$

12. Since $d \sec x = \text{tang } x \sec x\,dx$;

$$\therefore \int \text{tang } x \sec x\,dx = \sec x + C.$$

13. Since d cosec $x = - \cot x \text{ cosec } x\,dx$;

$$\therefore -\int \cot x \text{ cosec } x\,dx = \text{cosec } x + C.$$

14. Since d versin $x = \sin x\,dx$;

$$\therefore \int \sin x\,dx = \text{versin } x + C.$$

15. Since d coversin $x = - \cos x\,dx$;

$$\therefore -\int \cos x\,dx = \text{coversin } x + C.$$

16. Since $d \sin^{-1} x = \dfrac{dx}{\sqrt{1-x^2}}$;

$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$$

17. Since $d \cos^{-1} x = - \dfrac{dx}{\sqrt{1-x^2}}$;

$$\therefore -\int \frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C.$$

18. Since $d \text{ tang}^{-1} x = \dfrac{dx}{1+x^2}$;

$$\therefore \int \frac{dx}{1+x^2} = \text{tang}^{-1} x + C.$$

19. Since $d \cot^{-1} x = -\frac{dx}{1+x^2}$;

$$\therefore \quad -\int \frac{dx}{1+x^2} = \cot^{-1} x + C.$$

20. Since $d \sec^{-1} x = \frac{dx}{x\sqrt{x^2-1}}$;

$$\therefore \quad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C.$$

21. Since $d \operatorname{cosec}^{-1} x = -\frac{dx}{x\sqrt{x^2-1}}$;

$$\therefore \quad -\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{cosec}^{-1} x + C.$$

22. Since $d \operatorname{versin}^{-1} x = \frac{dx}{\sqrt{2x-x^2}}$;

$$\therefore \quad \int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1} x + C.$$

23. Since $d \operatorname{coversin}^{-1} x = -\frac{dx}{\sqrt{2x-x^2}}$;

$$\therefore \quad -\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{coversin}^{-1} x + C.$$

These twenty-three cases embrace all the known forms which can be directly integrated; and before any given function can be integrated, it must be brought, by algebraic processes, to correspond with some one of these known forms. It is in this algebraic reduction that the chief difficulty of the Integral Calculus consists; and it may be said, in fact, that the whole body of this Calculus is made up of artifices by means of which the simplification of complicated or unusual expressions may be effected.

NOTE.—We have seen that when $m = -1$, the fifth of the above formulas fails. For we then have

$$\int x^{-1}\,dx = \int \frac{dx}{x} = \frac{1}{0} = \infty,$$

whereas we know, from the sixth formula, that

$$\int \frac{dx}{x} = \log x + C.$$

This is, however, only an apparent failure; for if we write, as we are at liberty to do,

$$\int x^m\,dx = \frac{x^{m+1}}{m+1} + C = \frac{x^{m+1} - a^{m+1}}{m+1} + C',$$

we shall have, when $m = -1$,

$$\int x^{-1}\,dx = \frac{0}{0}.$$

The value of this $\frac{0}{0}$ may be found, as for indeterminate expressions, by differentiating the numerator and denominator of $\frac{x^{m+1} - a^{m+1}}{m+1}$ with respect to m, and putting $m = -1$ in the result.

We thus obtain

$$\int x^m\,dx = \frac{x^{m+1} - a^{m+1}}{m+1} + C' = \log x - \log a + C'$$
$$= \log x + C'', \text{ when } m = -1.$$

8. It will be observed that each of the above forms comes under the general form $\int F'(x)dx$, and it is evident that in this expression x may represent any variable, and therefore *any function of a variable.*

Wherefore, substituting $\phi(x)$ for (x), we have

$$\int F'\{\phi(x)\}d\,\phi(x) = F\{\phi(x)\} + C.$$

The expression under the integral sign is composed of two parts, viz:

(*a*), *a function of a function or variable;*

(*b*), *the differential of that function or variable;*

and whenever the given expression can be decomposed into two such parts, its integration may be effected at once by referring it to one of the fundamental forms. The following examples will serve for illustration:

1. $$\int(a + bx + cx^2)^2(b + 2cx)dx$$
$$= \int(a + bx + cx^2)^2\,d(a + bx + cx^2)$$
$$= \frac{(a + bx + cx^2)^3}{3} + C.$$

2. $$\int\frac{b + 2cx}{a + bx + cx^2}\,dx = \int\frac{d(a + bx + cx^2)}{a + bx + cx^2}$$
$$= \log\,(a + bx + cx^2) + C.$$

3. $$\int e^{2x}\,dx = \int\frac{1}{2}\,e^{2x}\,d(2x) = \frac{1}{2}\int e^{2x}\,d(2x)$$
$$= \frac{1}{2}\,e^{2x} + C.$$

4. $$\int 2\sin x\cos x\,dx = \int 2\sin x\,d\sin x$$
$$= \sin^2 x + C.$$

5. $$\int (a + bx^2)^3\, xdx = \int \frac{1}{2b}\,(a + bx^2)^3\, 2bxdx$$

$$= \frac{1}{2b}\int (a + bx^2)^3\, d(a + bx^2)$$

$$= \frac{1}{2b}\,\frac{(a + bx^2)^4}{4} + C.$$

9. Integration by Substitution.—When the expression $F'(x)dx$ can not be integrated directly, the operation may sometimes be effected by substituting for x some function of a new variable z.

Thus, if $x = \phi(z)$, then $dx = \phi'(z)dz$, and $F'(x) = F'\{\phi(z)\}$.

$$\therefore \quad \int F'(x)dx = \int F'\{\phi(z)\}\phi'(z)dz.$$

The relation between x and z being arbitrary, we may often choose it in such a manner as to render the integration comparatively easy; the only condition to be observed being that the definite integral with respect to z, *taken between any two limits* z_0 *and* Z, *shall be the same as that taken with respect to* x *between* the corresponding limits x_0 and X. We shall then have the following equation:

$$\int_{x_0}^{X} F'(x)\, dx = \int_{z_0}^{Z} F'\{\phi(z)\}\phi'(z)\, dz.$$

APPLICATION.

$$\int_{x_0}^{X} F(x + a)\, dx.$$

Put $x + a = z$; then $dx = dz$, and

$$\int_{x_0}^{X} F(x + a)\, dx = \int_{z_0}^{Z} F(z)\, dz = \int_{x_0 + a}^{X + a} F(z)\, dz$$

$$= \int_{x_0 + a}^{X + a} F(x + a)\, d(x + a).$$

10. Integration by Parts.—When the expression to be integrated is of the form udv, the integration may often be effected in the following manner:

Since $d(uv) = udv + vdu$, we have

$$uv = \int udv + \int vdu;$$

$$\therefore \quad \int udv = uv - \int vdu.$$

This is called the *formula for integration by parts.* It reduces the integration of udv to that of vdu, which may be simpler.

EXAMPLE.

$$\int \log x\, dx.$$

Put $\log x = u$, $dx = dv$; $\therefore$ $du = \dfrac{dx}{x}$, $v = x$, and

$$\int \log x\, dx = x \log x - \int x \frac{dx}{x} = x \log x - x + C.$$

11. Use of Constant Factors.—In effecting the reduction of a complicated expression it may often be advisable to introduce a constant factor. This may always be done provided that we take care to divide by this factor also; and since any constant factor within the integral sign may be placed before it as a coëfficient without affecting the *value* of the integral, it follows that *we may multiply the expression to be integrated by any constant factor, provided we place the reciprocal of that factor before the sign of integration as a coëfficient.*

There is an example of this operation in Ex. 5, Art. 8. As another example let us take the expression

$$\int (1 + 3x^4)^5 x^3\, dx.$$

We have $\int (1+3x^4)^5 x^3\,dx = \frac{1}{12}\int (1+3x^4)^5\, 12x^3\,dx$

$$= \frac{1}{12}\int (1+3x^4)^5 d(1+3x^4),$$

$$= \frac{1}{12}\,\frac{(1+3x^4)^6}{6} + C.$$

12. General Examples in Integration.

1. Integrate $(7x^5 + 6ax^2 + 6)dx$.

We have

$$\int (7x^5 + 6ax^2 + 6)dx = \int 7x^5\,dx + \int 6ax^2 dx + \int 6dx$$

$$= \frac{7}{6}\,x^6 + \frac{6}{3}\,ax^3 + 6x + C.$$

2. Integrate $(ax^n + b)^m x^{n-1}\,dx$.

We have

$$\int (ax^n + b)^m x^{n-1}\,dx = \frac{1}{an}\int (ax^n + b)^m\, anx^{n-1}\,dx$$

$$= \frac{1}{an}\int (ax^n + b)^m d(ax^n + b) = \frac{1}{an}\,\frac{(ax^n + b)^{m+1}}{m+1} + C.$$

3. Integrate $(2ax + x^2)^3 (a + x)dx$.

We have

$$\int (2ax + x^2)^3 (a + x)dx = \frac{1}{2}\int (2ax + x^2)^3 (2a + 2x)dx$$

$$= \frac{1}{2}\int (2ax + x^2)^3 d(2ax + x^2) = \frac{1}{2}\,\frac{(2ax + x^2)^4}{4} + C.$$

4. Integrate $\frac{x^2\,dx}{1+x^3}$.

We have $\int \frac{x^2\,dx}{1+x^3} = \frac{1}{3}\int \frac{3x^2\,dx}{1+x^3}$

$$= \frac{1}{3}\int \frac{d(1+x^3)}{1+x^3} = \frac{1}{3}\log(1+x^3) + C.$$

5. Integrate $\frac{a\,dx}{b+cx}$.

We have $\int \frac{a\,dx}{b+cx} = \frac{a}{c}\int \frac{c\,dx}{b+cx} = \frac{a}{c}\log(b+cx) + C,$

$$= \log(b+cx)^{\frac{a}{c}} + C.$$

6. Integrate $\frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx$.

Ans. $\log(x^3+x^2+x+1) + C.$

7. Integrate $\left\{\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{h}{x^4}\right\} dx$.

Ans. $a\log x - \frac{b}{x} - \frac{c}{2x^2} - \frac{h}{3x^3} + C.$

8. Integrate $\frac{7x\,dx}{8a-3x^2}$.

We have

$$\int \frac{7x\,dx}{8a-3x^2} = -\frac{7}{6}\int \frac{-6x\,dx}{8a-3x^2} = -\frac{7}{6}\log(8a-3x^2) + C,$$

$$= -\log(8a-3x^2)^{\frac{7}{6}} + \log c = \log\left\{\frac{c}{(8a-3x^2)^{\frac{7}{6}}}\right\},$$

by making $C = \log c$, which is evidently permissible, since $\log c$ is a constant.

9. Integrate $\dfrac{dx}{(x+a)^{\frac{1}{2}}+(x+b)^{\frac{1}{2}}}$.

Multiplying numerator and denominator by $(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}$, we have

$$\int = \int \frac{(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}}{a-b}\, dx$$

$$= \frac{1}{a-b}\int (x+a)^{\frac{1}{2}}\, dx - \frac{1}{a-b}\int (x+b)^{\frac{1}{2}}\, dx,$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right\} + C.$$

10. Integrate $\dfrac{dx}{(a+bx^2)^{\frac{3}{2}}}$.

We have

$$\int = \int (a+bx^2)^{-\frac{3}{2}}\, dx = -\frac{1}{2a}\int (ax^{-2}+b)^{-\frac{3}{2}}\,(-2ax^{-3}\, dx),$$

and by this transformation we have made the quantity in the second parenthesis the exact differential of that in the first. Integrating this last form, we have

$$\int = \frac{1}{a}(ax^{-2}+b)^{-\frac{1}{2}} + C = \frac{x}{a\sqrt{a+bx^2}} + C.$$

The method of transformation illustrated in this example may often be employed to advantage.

11. Integrate $\dfrac{dx}{(1-x^2)^{\frac{3}{2}}}$.

We have $\int = \int (1-x^2)^{-\frac{3}{2}}\, dx = \int (x^{-2}-1)^{-\frac{3}{2}}\, x^{-3}\, dx.$

Ans. $\dfrac{x}{\sqrt{1-x^2}} + C.$

12. Integrate $\dfrac{axdx}{(ax+bx^2)^{\frac{3}{2}}}$.

We have

$$\int = \int (ax+bx^2)^{-\frac{3}{2}} axdx = \int (ax^{-1}+b)^{-\frac{3}{2}} (x^2)^{-\frac{3}{2}} axdx,$$

$$= \int (ax^{-1}+b)^{-\frac{3}{2}} ax^{-2}\,dx = -\int (ax^{-1}+b)^{-\frac{3}{2}} (-ax^{-2}\,dx)$$

$$= 2(ax^{-1}+b)^{-\frac{1}{2}} = \frac{2x}{\sqrt{ax+bx^2}} + C.$$

13. Integrate $\dfrac{adx}{x\sqrt{bx+c^2x^2}}$.

We have

$$\int = a\int (bx+c^2x^2)^{-\frac{1}{2}} x^{-1}\,dx = \frac{a}{b}\int (bx^{-1}+c^2)^{-\frac{1}{2}} bx^{-2}dx.$$

$$\text{Ans.}\quad -\frac{2a}{b}\left\{\frac{bx+c^2x^2}{x^2}\right\}^{\frac{1}{2}} + C.$$

14. Integrate $(3ax^2+4bx^3)^{\frac{4}{3}}(2ax+4bx^2)\,dx$.

In order to integrate a differential function directly, it is necessary, as we have seen, that one of its factors shall be the differential of the expression of which the other factor is a function. It generally happens that this is not the case, but the second factor can often be brought to the proper form by introducing an unknown *constant factor* into the expression, and determining the value of this constant by some one of the ordinary algebraic methods.

In the present example, let us suppose that the multiplication of the second parenthesis by A will make it the differential of the first. Then we shall have

$$\int = \frac{1}{A}\int (3ax^2+4bx^3)^{\frac{4}{3}} (2Aax+4Abx^2)dx.$$

We must now determine A by the condition

$$d(3ax^2 + 4bx^3) = 2Aax\,dx + 4Abx^2\,dx.$$

But $$d(3ax^2 + 4bx^3) = (6ax + 12bx^2)dx;$$

$$\therefore\ 6ax + 12bx^2 = 2Aax + 4Abx^2;$$

and since this relation is independent of x, we must have

$$6a = 2Aa, \quad 12b = 4Ab,$$

from each of which equations we have $A = 3$.

Hence we have, finally,

$$\int = \frac{1}{3}\int (3ax^2 + 4bx^3)^{\frac{4}{3}} (6ax + 12bx^2)dx$$

$$= \frac{3}{21} (3ax^2 + 4bx^3)^{\frac{7}{3}} + C.$$

If the two values of A had not been the same, we would have inferred that there was no constant factor which would reduce the given expression to the required form.

15. $\int (4ax^3 + 5bx^2 + 6x)^4 (ax^2 + 2bx + 3)dx.$

Let A be the required factor. Then

$$\int = \frac{1}{A}\int (4ax^3 + 5bx^2 + 6x)^4 (Aax^2 + 2Abx + 3A)dx.$$

We must have

$$Aax^2 + 2Abx + 3A = 12ax^2 + 10bx + 6.$$

$$\therefore\ Aa = 12a, \text{ or } A = 12; \quad 2Ab = 10b, \text{ or } A = 5;$$

$$3A = 6, \text{ or } A = 2.$$

These values of A being inconsistent with each other, we infer that the given expression can not be integrated in this form.

16. Integrate $\tan 2x \sec 2x\, dx$.

We have $\int = \frac{1}{2}\int \tan 2x \sec 2x\, d2x = \sec 2x + C.$

17. Integrate $\sec^2 \sqrt{2x}\, x^{-\frac{1}{2}}\, dx$.

We have

$$\int = \int \sec^2 \sqrt{2x}\,(\sqrt{2}\, d\sqrt{2x}) = \sqrt{2}\tan\sqrt{2x} + C.$$

18. Integrate $\tan x\, dx$.

We have $\int = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{d \cos x}{\cos x} = -\log \cos x$

$$= \log \frac{1}{\cos x} = \log \sec x + C.$$

19. Integrate $\frac{dx}{a(1 + \cos x)}$.

We have

$$\int = \frac{1}{a}\int \frac{d\,\frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{1}{a}\int \sec^2 \frac{x}{2}\, d\,\frac{x}{2} = \frac{1}{a}\tan\frac{x}{2} + C.$$

20. Integrate $\sin^3 3x \cos 3x\, dx$.

We have

$$\int = \frac{1}{12}\int 4\sin^3 3x \cos 3x\, d3x = \frac{1}{12}\int 4\sin^3 3x\, d(\sin 3x)$$

$$= \frac{1}{12}\sin^4 3x + C.$$

21. Integrate $\frac{x+1}{\sqrt{1-x^2}}\, dx$.

We have

$$\int = \int \frac{x\, dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

22. Integrate $\frac{e^x\,dx}{1+e^{2x}}$. Ans. $\tan^{-1} e^x + C$.

23. Integrate $\frac{x^6\,dx}{1+x^2}$.

Dividing the numerator by the denominator, we have

$$\int = \int x^4\,dx - \int x^2\,dx + \int dx - \int \frac{dx}{1+x^2}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - \tan^{-1} x + C.$$

24. Integrate $\frac{6x\,dx}{\sqrt{4-9x^4}}$.

We have

$$\int = \int \frac{3x\,dx}{\sqrt{1-\frac{9}{4}x^4}} = \int \frac{d\left(\frac{3x^2}{2}\right)}{\sqrt{1-\left(\frac{3x^2}{2}\right)^2}} = \sin^{-1}\frac{3x^2}{2} + C.$$

25. Integrate $e^{\sec x}\sec^2 x \sin x\,dx$.

We have $\int = \int e^{\sec x} \sec x \tan x\,dx = e^{\sec x} + C$.

26. Integrate $\frac{dx}{\sqrt{a^2 - b^2x^2}}$.

We have $$\int = \int \frac{d\,\frac{x}{a}}{\sqrt{1-\frac{b^2x^2}{a^2}}} = \frac{1}{b}\int \frac{\frac{b}{a}\,dx}{\sqrt{1-\frac{b^2x^2}{a^2}}}$$

$$= \frac{1}{b}\int \frac{d\left(\frac{bx}{a}\right)}{\sqrt{1-\left(\frac{bx}{a}\right)^2}} = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C.$$

27. Integrate $\dfrac{dx}{a^2+b^2x^2}$.

We have $\displaystyle\int = \frac{1}{ab}\int \frac{d\,\frac{bx}{a}}{1+\left(\frac{bx}{a}\right)^2} = \frac{1}{ab}\operatorname{tang}^{-1}\frac{bx}{a} + C.$

28. Integrate $\dfrac{dx}{x\sqrt{b^2x^2-a^2}}$.

We have $\displaystyle\int = \frac{1}{a}\int \frac{d\,\frac{bx}{a}}{\frac{bx}{a}\sqrt{\left(\frac{bx}{a}\right)^2-1}} = \frac{1}{a}\sec^{-1}\frac{bx}{a} + C.$

29. Integrate $\dfrac{dx}{\sqrt{a^2x-b^2x^2}}$.

We have

$$\int = \frac{1}{b}\int \frac{\frac{2b^2}{a^2}\,dx}{\sqrt{2\left(\frac{2b^2x}{a^2}\right)-\left(\frac{2b^2x}{a^2}\right)^2}} = \frac{1}{b}\operatorname{versin}^{-1}\frac{2b^2x}{a^2} + C.$$

30. Integrate $\dfrac{dx}{1+x+x^2}$.

The denominator is equal to $\frac{3}{4}+\left(x+\frac{1}{2}\right)^2$.

$$\therefore \int = \int \frac{dx}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2} = \frac{4}{3}\int \frac{dx}{1+\left\{\frac{2x+1}{\sqrt{3}}\right\}^2}$$

$$= \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{d\left\{\frac{2x+1}{\sqrt{3}}\right\}}{1+\left\{\frac{2x+1}{\sqrt{3}}\right\}^2}$$

$$= \frac{2}{\sqrt{3}}\operatorname{tang}^{-1}\left\{\frac{2x+1}{\sqrt{3}}\right\} + C.$$

31. Integrate $\dfrac{\sqrt{x}\,dx}{\sqrt{2-4x^3}}$.

We have

$$\int = \frac{1}{2}\int \frac{\sqrt{2x}\,dx}{\sqrt{1-2x^3}} = \frac{1}{3}\int \frac{\frac{3}{2}\sqrt{2x}\,dx}{\sqrt{1-2x^3}} = \frac{1}{3}\sin^{-1}\sqrt{2x^3} + C.$$

32. Integrate $\dfrac{x^2\,dx}{\sqrt{2x^3-4x^6}}$.

We have

$$\int = \frac{1}{6}\int \frac{12x^2\,dx}{\sqrt{2(4x^3)-(4x^3)^2}} = \frac{1}{6}\,\text{versin}^{-1}(4x^3) + C.$$

33. Integrate $\dfrac{x^{-1}\,dx}{\sqrt{2x^{-3}-4}}$. Ans. $-\dfrac{1}{3}\sec^{-1}\sqrt{\dfrac{1}{2}x^{-3}} + C.$

34. Integrate $\dfrac{\frac{1}{2}x^{-1}\,dx}{\sqrt{\frac{1}{2}x^{-3}-1}}$.

We have $\displaystyle\int = \int \frac{1}{2}x^{-1}\left(\frac{1}{2}x^{-3}-1\right)^{-\frac{1}{2}}dx$

$$= -\frac{1}{3}\int \frac{-\frac{3}{4}x^{-4}\left(\frac{1}{2}x^{-3}-1\right)^{-\frac{1}{2}}dx}{1+\left(\frac{1}{2}x^{-3}-1\right)}$$

$$= -\frac{1}{3}\,\text{tang}^{-1}\sqrt{\frac{1}{2}x^{-3}-1} + C.$$

NOTE.—It will be observed that the last four examples are really the same under different forms.

35. Integrate $\dfrac{x\,dx}{\sqrt{a-bx^4}}$.

36. Integrate $\frac{x^{n-1}\,dx}{\sqrt{a^2 - b^2 x^{2n}}}$.

37. Integrate $\frac{dx}{x\sqrt{4x^4 - 3}}$.

38. Integrate $\frac{4dx}{x\sqrt{4x - x^2}}$.

39. Integrate $\frac{xdx}{\{2ax - x^2\}^{\frac{3}{2}}}$.

13. The foregoing examples will afford a sufficient illustration of direct integration. We pass now to the consideration of those cases in which the integration can not be directly effected, or in which it may be more readily effected by the methods of substitution, or decomposition into parts, or by development into series. We shall find it convenient to classify functions as *rational* or *irrational* algebraic functions, and *logarithmic*, *exponential*, and *trigonometric* functions.

CHAPTER III.

INTEGRATION OF RATIONAL FRACTIONS.

14. Every rational differential function of a variable may be considered as composed of two parts, one of which is an *entire function*, and the other is a *rational fraction* in which the degree of x in the numerator is less than in the denominator.

The entire function can always be integrated immediately by the processes already explained.

As for the fractional part, the method will be to decompose it into a series of partial fractions, each of which can be integrated separately.

The integration of rational fractions presents three cases:

1*st.* When the *factors* into which the *denominator* can be divided are *real* and *unequal.*

2*d.* When the factors of the denominator are *real* and *equal.*

3*d.* When the factors of the denominator are *imaginary.*

15. Case 1.—*Factors* of the *denominator real* and *unequal.*

Let $\frac{F(x)}{f(x)}dx$ be a rational fraction whose integral is to be determined, and let $a, b, c, \ldots . l$, be the values of x derived from the equation $f(x)=0$; then will $x-a$, $x-b$, $\ldots . x-l$, be the factors of $f(x)$. Assume

$$\frac{F(x)}{f(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\cdots\frac{L}{x-l} \quad (1),$$

in which A, B, C, etc., are unknown constants whose values may be found by the method of indeterminate coëfficients.

We have then

$$\frac{F(x)}{f(x)}dx=\frac{A\,dx}{x-a}+\frac{B\,dx}{x-b}+\cdots\frac{L\,dx}{x-l},$$

each term of which may be easily integrated.

It is obvious that if $F(x)$ be rational, the above assumption is legitimate, since, from equation (1), we have

$$\frac{F(x)}{f(x)}=\frac{A(x-b)(x-c)..(x-l)+B(x-a)(x-c)..(x-l)+..\text{etc.}}{f(x)},$$

and since this equation is entirely independent of the value of x, the values of A, B, C, etc., may be found by placing the coëfficients of the like powers of x in the two numerators equal to each other.

16. From equation (1) we have

$$F(x)=\frac{Af(x)}{x-a}+\frac{Bf(x)}{x-b}+\frac{Cf(x)}{x-c}+\text{etc.}\quad\ldots\ldots\quad(2).$$

If in (2) we make $x=a$, all the terms in the second member will reduce to zero, except the first, and we shall have

$$F(a)=\frac{Af(a)}{x-a}.$$

But when $x=a$, $f(x)=f(a)=0$, and $x-a=0$; whence

$$\frac{f(a)}{x-a}=\frac{0}{0}=f'(a),$$

according to the methods established for finding the values of functions which reduce to the form $\frac{0}{0}$.

Therefore we have

$$F(a)=Af'(a),\ \text{and}\ A=\frac{F(a)}{f'(a)}.$$

In the same manner we have

$$F(b)=Bf'(b),\ \text{and}\ B=\frac{F(b)}{f'(b)};$$

$$F(c)=Cf'(c),\ \text{and}\ C=\frac{F(c)}{f'(c)};\ \text{and so on.}$$

This method of finding the values of A, B, C, etc., will usually be preferable to that of indeterminate coëfficients.

Examples.

1. Integrate $\dfrac{a\,dx}{x^2+bx}$.

Assume $\dfrac{a}{x^2+bx}=\dfrac{A}{x}+\dfrac{B}{x+b}$.

Then $$\frac{a}{x^2+bx}=\frac{(A+B)x+Ab}{x^2+bx};$$

$$\therefore\ a=(A+B)x+Ab;$$

$$\therefore\ A+B=0,\text{ and } B=-A;$$

$$Ab=a,\text{ and } A=\frac{a}{b},\text{ whence } B=-\frac{a}{b}.$$

Substituting these values of A and B, we have

$$\int\frac{a\,dx}{x^2+bx}=\frac{a}{b}\int\frac{dx}{x}-\frac{a}{b}\int\frac{dx}{x+b}$$

$$=\frac{a}{b}\{\log x-\log(x+b)\}$$

$$=\frac{a}{b}\log\frac{x}{x+b}+C,$$

$$=\log\left\{\frac{cx}{x+b}\right\}^{\frac{a}{b}}.$$

2. Integrate $\dfrac{a\,dx}{x^2+bx}$ by the second method.

We have $F(x)=a$, $f(x)=x^2+bx$; whence $x=0$, or $x=-b$.

$$\therefore\ A=\frac{F(a)}{f'(a)}=\frac{a}{b}\text{ and } B=\frac{F(b)}{f'(b)}=-\frac{a}{b},$$

$\therefore$ as in the preceding example,

$$\int=\frac{a}{b}\log x-\frac{a}{b}\log(x+b)+C=\log\left\{\frac{cx}{x+b}\right\}^{\frac{a}{b}}.$$

17. Case II.—*The factors* of the *denominator real* and *equal.*

Let $\frac{F(x)}{f(x)}dx$ be a rational fraction whose denominator contains m factors equal to $x - a$, and assume

$$f(x) = (x - a)^m \phi(x).$$

If we take $$\frac{F(x)}{f(x)} = \frac{A}{(x-a)^m} + \frac{B}{(x-a)^{m-1}}$$

$$+ \dots \frac{L}{x-a} + \frac{P}{\phi(x)} \quad (1),$$

the values of A, B, . . . P, may be determined as by the first method in the preceding case, and the original fraction will be thus decomposed into a series of partial fractions, each of which is integrable by the ordinary methods.

The values of A, B, C, etc., may, however, be determined more elegantly in the following manner:

We have, from (1),

$$F(x) = A\phi(x) + B(x - a)\phi(x) + \dots P(x - a)^m \quad (2).$$

Making $x = a$ in this equation, we have

$$F(a) = A\phi(a), \quad \therefore A = \frac{F(a)}{\phi(a)}.$$

Differentiating (2), and making $x = a$, we have

$$F'(a) = A\phi'(a) + B\phi(a),$$

from which B may be found.

Differentiating (2) $m - 1$ times in succession, and after each differentiation making $x = a$ in the result, we introduce at each step a new constant whose value may be obtained at once in terms of known quantities.

The general equations from which the constants are to be found are the following:

$$\left.\begin{aligned}
F(a) &= A\,\phi(a),\\
F'(a) &= A\,\phi'(a) + B\,\phi(a),\\
F''(a) &= A\,\phi''(a) + 2B\phi'(a) + 2.\,1\;C\phi(a),\\
F'''(a) &= A\,\phi'''(a) + 3\,B\phi''(a)\\
&\qquad + 3.\,2.\;C\phi'(a) + 3.\,2.\,1.\;D\,\phi(a),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
F^{m-1}(a) &= A\,\phi^{m-1}(a) + (m-1)\;B\,\phi^{m-2}(a)\\
&\qquad + (m-1)\;(m-2)C\phi^{m-3}(a) +\\
\dots\dots &\quad + (m-1)(m-2)(m-3)\;..\;3.2.1\;L\phi(a)
\end{aligned}\right\}\;(3).$$

The constants corresponding to the multiple factor $x-a$ having been determined, we may operate in the same manner upon the fraction $\dfrac{P}{\phi(x)}$, and so on, until the original fraction has been decomposed into all the parts of which it is susceptible.

NOTE.—The mode of decomposition adopted in this case is the only admissible one; for if we had pursued the method employed in Case I, the number of independent equations obtained would have been less than the number of constants, A, B, etc., to be determined.

EXAMPLES.

1. Integrate $\dfrac{x^2-4x+3}{x^3-6x^2+9x}\,dx$. We have

$$F(x) = x^2 - 4x + 3;\quad f(x) = x^3 - 6x^2 + 9x = (x-3)^2 x.$$

Assuming $\dfrac{x^2-4x+3}{x^3-6x^2+9x} = \dfrac{A}{(x-3)^2} + \dfrac{B}{x-3} + \dfrac{C}{x}$,

reducing to a common denominator, and placing the numerators equal to each other, we have

$$x^2 - 4x + 3 = Ax + Bx^2 - 3Bx + Cx^2 - 6Cx + 9C.$$

$$\therefore\ B + C = 1;\quad A - 3B - 6C = -4;\quad 9C = 3.$$

$$\therefore\ C = \frac{1}{3};\quad B = \frac{2}{3};\quad A = 0.$$

$$\therefore \int = \frac{2}{3}\int\frac{dx}{x-3} + \frac{1}{3}\int\frac{dx}{x}$$

$$= \frac{2}{3}\log(x-3) + \frac{1}{3}\log x + \log c.$$

$$= \log\{cx(x-3)^2\}^{\frac{1}{3}}.$$

2. The same by the second method.

We have $F(x) = x^2 - 4x + 3;\ f(x) = (x-3)^2 x.$

$$\therefore\ a = 3;\quad \phi(x) = x;\quad F(a) = 0;\quad \phi(a) = 3.$$

$$\therefore\ A = \frac{F(a)}{\phi(a)} = 0;\quad B = \frac{F'(a) - A\phi'(a)}{\phi(a)} = \frac{2}{3}.$$

Substituting these values of A and B in formula (2), and reducing, we have $P = \frac{1}{3}$.

$$\therefore\ \frac{F(x)}{f(x)} = \frac{2}{3(x-3)} + \frac{1}{3x},$$

and, as in the previous case,

$$\int = \frac{2}{3}\int\frac{dx}{x-3} + \frac{1}{3}\int\frac{dx}{x} = \log\{cx(x-3)^2\}^{\frac{1}{3}}.$$

18. Case III.—*The factors* of the denominator *imaginary.*

When the denominator of the proposed fraction contains imaginary factors, the method of decomposition may be entirely similar to the foregoing.

Since, however, for every factor of the form

$$x - a - b\sqrt{-1},$$

there must be another factor of the form

$$x - a + b\sqrt{-1}$$

(otherwise the product of the factors would not be real), it follows that for every partial fraction of the form

$$\frac{M - N\sqrt{-1}}{x - a - b\sqrt{-1}}$$

there will also be one of the form

$$\frac{M + N\sqrt{-1}}{x - a + b\sqrt{-1}}.$$

If we add these two fractions together, the imaginary terms will disappear, and we shall have their sum,

$$\frac{2M(x-a) + 2bN}{(x-a)^2 + b^2},$$

which may be written

$$\frac{Ax + B}{(x-a)^2 + b^2},$$

and this will be the form of the partial fraction corresponding to the two imaginary factors $x - a \mp b\sqrt{-1}$, or to the single equivalent quadratic factor $(x-a)^2 + b^2$.

To integrate $\dfrac{Ax + B}{(x-a)^2 + b^2}\,dx$, assume

$$\int = \int \frac{A(x-a)\,dx}{(x-a)^2 + b^2} + \int \frac{Aa + B}{(x-a)^2 + b^2}\,dx.$$

The integral of the first term is evidently equal to

$$\frac{1}{2} A \log \{(x-a)^2 + b^2\},$$

and that of the second term is easily seen to be equal to

$$\frac{Aa+B}{b} \operatorname{tang}^{-1} \frac{x-a}{b}.$$

18′. If the denominator should contain n *equal quadratic* factors, then we should have, by Case II,

$$\frac{F(x)}{f(x)} = \frac{Ax+B}{\{(x-a)^2+b^2\}^n} + \frac{Cx+D}{\{(x-a)^2+b^2\}^{n-1}};$$

$$+ \ldots \ldots \frac{Lx+M}{(x-a)^2+b^2};$$

and the general term to be integrated in such cases is

$$\frac{Ax+B}{\{(x-a)^2+b^2\}^n} dx.$$

To integrate this expression, assume

$$\int = \int \frac{A(x-a)\,dx}{\{(x-a)^2+b^2\}^n} + \int \frac{Aa+B}{\{(x-a)^2+b^2\}^n} dx.$$

We easily find the integral of the first term to be

$$-\frac{A}{2(n-1)\{(x-a)^2+b^2\}^{n-1}};$$

and we have now to find that of the second term.

Let $x-a=bz$, and the second term may be written

$$\frac{Aa+B}{b^{2n-1}} \int \frac{dz}{(1+z^2)^n};$$

so that the operation is reduced to finding the integral of this last expression.

Now we have

$$\frac{1}{(1+z^2)^n}=\frac{1+z^2-z^2}{(1+z^2)^n}=\frac{1+z^2}{(1+z^2)^n}-\frac{z^2}{(1+z^2)^n}$$

$$=\frac{1}{(1+z^2)^{n-1}}-\frac{z^2}{(1+z^2)^n}.$$

$$\therefore \int\frac{dz}{(1+z^2)^n}=\int\frac{dz}{(1+z^2)^{n-1}}-\int\frac{z^2\,dz}{(1+z^2)^n}.$$

The last term of this equation may be integrated by the formula for integration by parts, viz:

$$\int udv=uv-\int vdu.$$

Let $z=u$ and $\dfrac{zdz}{(1+z^2)^n}=dv.$

Then $v=-\dfrac{1}{2(n-1)(1+z^2)^{n-1}}$ and $du=dz.$

$$\therefore \int udv=\int\frac{z^2\,dz}{(1+z^2)^n}=-\frac{z}{2(n-1)(1+z^2)^{n-1}}$$

$$+\frac{1}{2(n-1)}\int\frac{dz}{(1+z^2)^{n-1}};$$

and $$\int\frac{dz}{(1+z^2)^n}=\frac{z}{2(n-1)(1+z^2)^{n-1}}$$

$$+\left\{1-\frac{1}{2(n-1)}\right\}\int\frac{dz}{(1+z^2)^{n-1}}.$$

We have thus reduced the integration of the original expression to that of $\dfrac{dz}{(1+z^2)^{n-1}}$, in which the exponent of the denominator is one less than the original exponent.

The continued application of the formula for integration by parts, reducing the exponent of the denominator by unity, at each step, will finally bring us to the form

$$\int \frac{dz}{1+z^2} = \text{tang}^{-1} z = \text{tang}^{-1} \frac{x-a}{b} + C;$$

and thus the original fraction is completely integrated.

19. Examples of Rational Fractions.

1. Integrate $\dfrac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6} dx.$

Placing the denominator equal to zero, we find the values of x to be 1, 2, and 3, and therefore the factors of the denominator are

$$x-1, \quad x-2, \quad x-3.$$

Assume

$$\frac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6} = \frac{F(x)}{f(x)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Then $A = \dfrac{F(1)}{f'(1)} = \dfrac{1 - 7 + 1}{3(1)^2 - 12(1) + 11} = -\dfrac{5}{2};$

$$B = \frac{F(2)}{f'(2)} = 9;$$

$$C = \frac{F(3)}{f'(3)} = -\frac{11}{2}.$$

$$\therefore \int = -\frac{5}{2}\int \frac{dx}{x-1} + 9\int \frac{dx}{x-2} - \frac{11}{2}\int \frac{dx}{x-3}$$

$$= -\frac{5}{2}\log(x-1) + 9\log(x-2) - \frac{11}{2}\log(x-3) + C$$

$$= \log.\frac{\{c(x-2)\}^9}{\sqrt{(x-1)^5(x-3)^{11}}}.$$

2. Integrate $\dfrac{2x-5}{(x+3)(x+1)^2}\,dx$.

We have $F(x)=2x-5$; $f(x)=(x+1)^2(x+3)$;

$\therefore$ by Case II, $a=-1$; $n=2$; $\phi(x)=x+3$.

Assume

$$\frac{2x-5}{(x+3)(x+1)^2}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{P}{x+3} \quad (1).$$

Then $F(a)=A\phi(a)$, $\therefore A=-\dfrac{7}{2}$;

$$F'(a)=A\phi'(a)+B\phi(a) \quad \therefore B=\frac{11}{4}.$$

If we suppose $a=-3$, and $\phi(x)=(x+1)^2$, we shall find $A=-\dfrac{11}{4}$, and this will evidently be the value of P in (1).

$$\therefore \int=-\frac{7}{2}\int\frac{dx}{(x+1)^2}+\frac{11}{4}\int\frac{dx}{x+1}-\frac{11}{4}\int\frac{dx}{x+3}$$

$$=\frac{7}{2}\,\frac{1}{x+1}+\frac{11}{4}\log\frac{x+1}{x+3}+\log C.$$

3. Integrate $\dfrac{5x-2}{x^3+6x^2+8x}\,dx$.

The factors of the denominator are

$$x,\ x+2,\ x+4.$$

$$\therefore \frac{5x-2}{x^3+6x^2+8x}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+4}.$$

$$\text{Ans. } \int=\frac{1}{4}\log\left\{\frac{c(x+2)^{12}}{x(x+4)^{11}}\right\}.$$

4. Integrate $\dfrac{x\,dx}{(x+2)(x+3)^2}$.

$$\text{Ans. } \int = -\frac{3}{x+3} + \log\left\{\frac{c(x+3)}{x+2}\right\}^2.$$

5. Integrate $\dfrac{x^2+3x+1}{x^3+x^2-2x}\,dx$.

$$\text{Ans. } \int = \frac{1}{3}\log\frac{c(x-1)^5}{x\sqrt{x^2+2x}}.$$

6. Integrate $\dfrac{x^2\,dx}{(x+2)^2(x+4)^2}$.

We have $F(x)=x^2$; $f(x)=(x+2)^2(x+4)^2$;

$$a=-2;\quad n=2;\quad \phi(x)=(x+4)^2.$$

$$\text{Ans. } \int = -\frac{5x+12}{x^2+6x+8} + \log\left\{\frac{c(x+4)}{x+2}\right\}^2.$$

7. Integrate $\dfrac{2x\,dx}{(x^2+1)(x^2+3)}$.

This will be integrated by Case III, since the real factors of the denominator are quadratic.

Assuming $\dfrac{2x}{(x^2+1)(x^2+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+3)}$,

we find $A=1$, $B=0$, $C=-1$, $D=0$.

$$\therefore \int = \int\frac{x\,dx}{x^2+1} - \int\frac{x\,dx}{x^2+3} = \log c\sqrt{\frac{x^2+1}{x^2+3}}.$$

8. Integrate $\dfrac{x^2\,dx}{(x^2+1)(x^2+4)}$.

We shall find $\int = \frac{4}{3}\int\frac{dx}{x^2+4} - \frac{1}{3}\int\frac{dx}{x^2+1}$

$$= \frac{2}{3}\int\frac{d\left(\frac{x}{2}\right)}{1+\left(\frac{x}{2}\right)^2} - \frac{1}{3}\int\frac{dx}{1+x^2}$$

$$= \frac{1}{3}\left\{2 \operatorname{tang}^{-1}\frac{x}{2} - \operatorname{tang}^{-1} x\right\} + C.$$

9. Integrate $\frac{x\,dx}{(x+1)(x+2)(x^2+1)}$.

Assuming

$$\frac{x}{(x+1)(x+2)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1},$$

we find $\int = -\frac{1}{2}\int\frac{dx}{x+1} + \frac{2}{5}\int\frac{dx}{x+2}$

$$+ \frac{1}{10}\int\frac{x\,dx}{x^2+1} + \frac{3}{10}\int\frac{dx}{x^2+1}$$

$$= -\frac{1}{2}\log(x+1) + \frac{2}{5}\log(x+2)$$

$$+ \frac{1}{20}\log(x^2+1) + \frac{3}{10}\operatorname{tang}^{-1} x + C.$$

10. Integrate $\frac{x\,dx}{(x+1)(x+2)(x^2+3)}$.

Ans. $\int = \frac{1}{28}\log\left\{\frac{(x+2)^8}{(x+1)^7\sqrt{x^2+3}} + 3\sqrt{3}\operatorname{tang}^{-1}\frac{x}{\sqrt{3}}\right\} + C.$

11. Integrate $\frac{dx}{x^3-1}$.

Assume $\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

$$\therefore \quad A = \frac{1}{3}; \quad B = -\frac{1}{3}; \quad C = -\frac{2}{3}; \quad \text{and}$$

$$\int = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx.$$

The Integral of the first term is $\frac{1}{3}\log(x-1)$.

To integrate the second term assume

$$x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+\frac{3}{4} = z^2+\frac{3}{4}; \text{ whence,}$$

$$x+\frac{1}{2} = z, \; dx = dz, \text{ and } x+2 = z+\frac{3}{2}.$$

$$\therefore \quad \frac{1}{3}\int \frac{(x+2)dx}{x^2+x+1}$$

$$= \frac{1}{3}\int \frac{\left(z+\frac{3}{2}\right)dz}{z^2+\frac{3}{4}} = \frac{1}{3}\int \frac{z\,dz}{z^2+\frac{3}{4}} + \frac{1}{3}\int \frac{\frac{3}{2}\,dz}{z^2+\frac{3}{4}}$$

$$= \frac{1}{6}\int \frac{2z\,dz}{z^2+\frac{3}{4}} + \frac{1}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}\,dz}{1+\frac{4z^2}{3}}$$

$$= \frac{1}{6}\log\left\{z^2+\frac{3}{4}\right\} + \frac{1}{\sqrt{3}}\text{tang}^{-1}\frac{2z}{\sqrt{3}} + C.$$

$$\therefore \text{ the whole integral} = \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1)$$

$$- \frac{1}{\sqrt{3}}\text{tang}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

12. Integrate $\frac{dx}{1-x^6}$.

The factors of the denominator are

$$1+x, \quad 1-x, \quad 1+x+x^2, \quad 1-x+x^2.$$

$\therefore$ assuming

$$\frac{1}{1-x^6} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{Cx+D}{1+x+x^2} + \frac{Ex+F}{1-x+x^2},$$

we find $\quad A = \frac{1}{6}; \quad B = \frac{1}{6}; \quad C = \frac{1}{6};$

$$D = \frac{1}{3} = F; \quad E = -\frac{1}{6}.$$

$$\therefore \int = \frac{1}{6}\int\frac{dx}{1+x} + \frac{1}{6}\int\frac{dx}{1-x} + \frac{1}{6}\int\frac{x+2}{1+x+x^2}\,dx$$

$$-\frac{1}{6}\int\frac{x-2}{1-x+x^2}\,dx$$

$$= \frac{1}{6}\log\left(\frac{1+x}{1-x}\right) + \frac{1}{12}\int\frac{2x+1}{1+x+x^2}\,dx + \frac{1}{12}\int\frac{3\,dx}{1+x+x^2}$$

$$-\frac{1}{12}\int\frac{2x-1}{1-x+x^2}\,dx + \frac{1}{12}\int\frac{3\,dx}{1-x+x^2}$$

$$= \frac{1}{6}\log\left(\frac{1+x}{1-x}\right) + \frac{1}{12}\log\left\{\frac{1+x+x^2}{1-x+x^2}\right\}$$

$$+\frac{1}{2\sqrt{3}}\int\frac{\frac{2}{\sqrt{3}}\,dx}{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2}$$

$$+\frac{1}{2\sqrt{3}}\int\frac{\frac{2}{\sqrt{3}}\,dx}{1+\frac{4}{3}\left(x-\frac{1}{2}\right)^2}$$

$$= \log \sqrt[12]{\frac{(1 + 2x + x^2)(1 + x + x^2)}{(1 - 2x + x^2)(1 - x + x^2)}}$$

$$+ \frac{1}{2\sqrt{3}} \left\{ \tang^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tang^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right\} + C.$$

13. Integrate $\frac{x^4\, dx}{2 + 3x^2}$.

$$\text{Ans. } \int = \frac{x^3}{9} - \frac{2x}{9} + \frac{2}{9}\sqrt{\frac{2}{3}} \tang^{-1} x \sqrt{\frac{3}{2}} + C.$$

14. Integrate $\frac{1 - x + x^2}{1 + x + x^2 + x^3}\, dx$.

$$\text{Ans. } \int = \frac{1}{2}\left\{ \log \frac{(1+x)^3}{\sqrt{1+x^2}} - \tang^{-1} x \right\} + C.$$

15. Integrate $\frac{dx}{x^4 + 4x + 3}$.

$$\text{Ans. } \int = -\frac{1}{6(x+1)} + \frac{1}{9} \log\left(\frac{x+1}{\sqrt{x^2 - 2x + 3}}\right)$$

$$+ \frac{1}{18\sqrt{2}} \tang^{-1} \frac{x-1}{\sqrt{2}} + C.$$

16. Integrate $\frac{x^2\, dx}{1 + x^4}$.

$$\text{Ans. } \int = \frac{1}{4\sqrt{2}} \log\left(\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right)$$

$$+ \frac{1}{2\sqrt{2}} \tang^{-1} \frac{x\sqrt{2}}{1 - x^2} + C.$$

CHAPTER IV.

INTEGRATION OF IRRATIONAL FUNCTIONS.

20. The number of irrational functions whose integrals have been determined in finite terms, is comparatively small, and in all cases the object has been effected by transforming the given irrational function into an equivalent rational function of another variable.

The case most frequently presented may be expressed, in general terms, as follows:

$$F\{x, \sqrt{a+bx+x^2}\}\,dx;$$

and the object is to rationalize the radical part of this expression.

Case I. — (*a.*) If the term containing x^2 has the positive sign, assume

$$\sqrt{a+bx+x^2} = z + x \qquad (1).$$

Then, $a + bx = z^2 + 2zx;\quad x = \dfrac{z^2 - a}{b - 2z};$

$$dx = -\frac{2(z^2 - bz + a)}{(b-2z)^2}\,dz;$$

$$\sqrt{a+bx+x^2} = \frac{z^2 - bz + a}{2z - b}.$$

These values, substituted in the original expression, will reduce it to a rational function of z, of the same character with F.

We will then integrate with respect to z, and in the result substitute for z its value derived from (1), viz:

$$z = \sqrt{a+bx+x^2} - x;$$

which operation will evidently give us the integral in terms of x.

(*b.*) If a is not negative, assume

$$\sqrt{a+bx+x^2}=\sqrt{a}+xz;$$

whence $b+x=2z\sqrt{a}+xz^2$; $\quad x=\dfrac{2z\sqrt{a}-b}{1-z^2}$;

$$dx=2\,\frac{z^2\sqrt{a}-bz+\sqrt{a}}{(1-z^2)^2}\,dz;$$

$$\sqrt{a+bx+x^2}=\frac{z^2\sqrt{a}-bz+\sqrt{a}}{1-z^2}.$$

The substitution of these values in the original function will render it rational in terms of z.

(*c.*) If the expression $a+bx+x^2$ can be decomposed into two binomial factors, $x-\alpha$, $x-\beta$, assume

$$\sqrt{a+bx+x^2}=(x-\alpha)z;$$

whence $a+bx+x^2=(x-\alpha)(x-\beta)=(x-\alpha)^2z^2$;

$$x-\beta=(x-\alpha)z^2;\quad x=\frac{\beta-\alpha z^2}{1-z^2};$$

$$dx=\frac{2z(\beta-\alpha)}{(1-z^2)^2}\,dz;\quad \sqrt{a+bx+x^2}=\frac{(\beta-\alpha)z}{1-z^2}.$$

These values, substituted in the original function, will render it rational in terms of z.

Case II.—If x^2 be negative, we may make use of the second or third of the preceding transformations. The first can not be used in this case, since it will not rationalize the given radical, nor can the second be used if a be negative.

21. If an irrational function contains two radicals of the forms $\sqrt{a+x}$, $\sqrt{b+x}$, it may be easily rationalized thus:

Let $\sqrt{a+x}=z$; whence

$$x=z^2-a,\quad dx=2z\,dz;\quad \sqrt{b+x}=\sqrt{b+z^2-a}.$$

This reduces the two radicals to a single one involving z, to which the methods of the last article may be applied.

22. When the only irrational quantities contained in the given expression are of the forms

$$\{f(x)\}^{\frac{m}{n}},\quad \{f(x)\}^{\frac{p}{q}},\quad \{f(x)\}^{\frac{r}{s}},\ \text{etc.},$$

they may be rationalized by assuming $f(x)=z^{nqs}$; whence

$$\{f(x)\}^{\frac{m}{n}}=z^{mqs};\quad \{f(x)\}^{\frac{p}{q}}=z^{nps};\ \text{etc.};$$

and the given expression is thus rendered rational in terms of z.

23. APPLICATIONS.

1. Integrate $\dfrac{dx}{\sqrt{a+bx+x^2}}$.

Assume $\sqrt{a+bx+x^2}=z-x$.

Then $a+bx=z^2-2xz$; $x=\dfrac{z^2-a}{b+2z}$;

$$dx=\frac{2(z-x)dz}{b+2z};\quad \frac{dx}{z-x}=\frac{dx}{\sqrt{a+bx+x^2}}=\frac{2dz}{b+2z}.$$

$$\therefore \int=\int\frac{2\,dz}{b+2z}=\int\frac{dz}{\frac{b}{2}+z}=\log\left\{\frac{b}{2}+z\right\}$$

$$=\log\left\{\frac{b}{2}+x+\sqrt{a+bx+x^2}\right\}+C.$$

If $b = 0$ we have

$$\int \frac{dx}{\sqrt{a + x^2}} = \log \{x + \sqrt{a + x^2}\} + C;$$

and if $a = 1$, then

$$\int \frac{dx}{\sqrt{1 + x^2}} = \log \{x + \sqrt{1 + x^2}\} + C.$$

These two last integrals are of very frequent occurrence.

2. Integrate $\dfrac{dx}{\sqrt{a + bx - x^2}}$.

Assume $\qquad \sqrt{a + bx - x^2} = \sqrt{a} + xz.$

Then $\qquad b - x = 2z\sqrt{a} + xz^2;$

$$\frac{dx}{\sqrt{a} + xz} = \frac{dx}{\sqrt{a + bx - x^2}} = -\frac{2\,dz}{1 + z^2}.$$

$$\therefore \int = -2\int \frac{dz}{1 + z^2} = -2 \operatorname{tang}^{-1} z$$

$$= -2 \operatorname{tang}^{-1} \left\{ \frac{\sqrt{a + bx - x^2} - \sqrt{a}}{x} \right\} + C.$$

3. Integrate $\dfrac{dx}{\sqrt{a + bx - x^2}}$,

under the supposition that the denominator contains two real binomial factors.

Assume $\qquad \sqrt{a + bx - x^2} = (x - \alpha)z,$ and

$$x^2 - bx - a = (x - \alpha)(x - \beta).$$

Then $\beta - x = (x - \alpha)z^2; \quad dx(1 + z^2) = -2(x - \alpha)z\,dz;$

$$\frac{dx}{(x - \alpha)z} = -\frac{2\,dz}{1 + z^2}.$$

$$\therefore \int = -2\int \frac{dz}{1+z^2} = -2 \text{ tang}^{-1} z$$

$$= -2 \text{ tang}^{-1} \sqrt{\frac{\beta - x}{x - a}} + C;$$

and, substituting for $\beta - x$, $x - a$, their values found by solving the equation $x^2 - bx - a = 0$, we have

$$\int \frac{dx}{\sqrt{a + bx - x^2}} = -2 \text{ tang}^{-1} \sqrt{\frac{b - 2x + \sqrt{b^2 + 4a}}{2x - b + \sqrt{b^2 + 4a}}} + C.$$

If $b = 0$, we have

$$\int \frac{dx}{\sqrt{a - x^2}} = -2 \text{ tang}^{-1} \sqrt{\frac{\sqrt{a} - x}{\sqrt{a} + x}} + C;$$

and if $a = 1$, then

$$\int \frac{dx}{\sqrt{1 - x^2}} = -2 \text{ tang}^{-1} \sqrt{\frac{1 - x}{1 + x}} + C.$$

4. Integrate $\dfrac{\alpha x + \beta}{\sqrt{a + bx - x^2}} dx.$

Assume

$$\frac{\alpha x + \beta}{\sqrt{a + bx - x^2}} = \frac{\alpha\left(x - \frac{b}{2}\right)}{\sqrt{a + bx - x^2}} + \frac{\beta + \frac{\alpha b}{2}}{\sqrt{a + bx - x^2}}.$$

Then we shall have

$$\int = \alpha \int \frac{x - \frac{b}{2}}{\sqrt{a + bx - x^2}} dx + \left(\beta + \frac{\alpha b}{2}\right) \int \frac{dx}{\sqrt{a + bx - x^2}}.$$

The integral of the first term is $-\alpha\sqrt{a + bx - x^2}$, and that of the second term is given in the previous examples.

5. Integrate $\frac{dx}{x\sqrt{a + bx \pm x^2}}$.

Assume $x = \frac{1}{z}$. Then $dx = -\frac{dz}{z^2}$, and

$$\frac{dx}{x\sqrt{a + bx \pm x^2}} = -\frac{dz}{z\sqrt{a + \frac{b}{z} \pm \frac{1}{z^2}}}$$

$$= -\frac{dz}{\sqrt{az^2 + bz \pm 1}},$$

the integral of which can be obtained as in the previous examples.

6. Integrate $\frac{dx}{x^2\sqrt{a + bx \pm x^2}}$.

Assuming $x = \frac{1}{z}$, we have $dx = -\frac{dz}{z^2}$.

$$\therefore \int = -\int \frac{dz}{\sqrt{a + \frac{b}{z} \pm \frac{1}{z^2}}} = -\int \frac{z\,dz}{\sqrt{az^2 + bz \pm 1}}$$

$$= -\frac{1}{a}\int \frac{az + \frac{b}{2} - \frac{b}{2}}{\sqrt{az^2 + bz \pm 1}}\,dz$$

$$= -\frac{1}{a}\int \frac{az + \frac{b}{2}}{\sqrt{az^2 + bz \pm 1}}\,dz + \frac{b}{2a}\int \frac{dz}{\sqrt{az^2 + bz \pm 1}}$$

$$= -\frac{1}{a}\sqrt{az^2 + bz \pm 1} + \frac{b}{2a}\int \frac{dz}{\sqrt{az^2 + bz \pm 1}},$$

the last term of which is integrable as in the preceding examples.

Binomial Differentials.

24. The remaining irrational forms which admit of integration in finite terms belong, for the most part, to the class of *binomial* differentials, which may be written

$$x^m(a+bx^n)^p\,dx.$$

In this expression m and n are either whole numbers, or they may be made integrals by the methods of Art. 22, so that the only necessary fractional exponent is p.

[1]. To integrate this expression,

$$\text{assume} \quad (a+bx^n)^p = z^p \qquad (1);$$

$$\text{then} \qquad x = \left\{\frac{z-a}{b}\right\}^{\frac{1}{n}};$$

$$dx = \frac{1}{nb}\left\{\frac{z-a}{b}\right\}^{\frac{1}{n}-1} dz \;. \quad (2);$$

$$x^m = \left\{\frac{z-a}{b}\right\}^{\frac{m}{n}} \qquad (3).$$

Multiplying (1), (2), and (3), together, we have

$$x^m(a+bx^n)^p\,dx = \frac{1}{nb}\,z^p\left\{\frac{z-a}{b}\right\}^{\frac{m+1}{n}-1} dz \qquad (4).$$

If $\dfrac{m+1}{n}$ be a *whole number*, the only irrational quantity in the second member of equation (4) is z^p, and this may be rationalized by Art. 22. Thus if $p = \dfrac{r}{q}$ we may assume $z = t^q$, which is the same thing as taking $a + bx^n = t^q$; and by pursuing the method above illustrated, we may easily rationalize and integrate the given expression in terms of t.

[2]. The given expression may be written

$$x^m(a+bx^n)^p\,dx = x^{m+np}(ax^{-n}+b)^p\,dx.$$

If we put $ax^{-n} + b = z$, then, according to the preceding case, the expression can be integrated whenever

$$\frac{m + np + 1}{-n} = -\left\{\frac{m+1}{n} + p\right\} \text{ is a whole number.}$$

Hence there are two cases in which a binomial differential can be rationalized.

1st. *When the exponent of the variable without the parenthesis, increased by unity, is exactly divisible by the exponent of the variable within the parenthesis.*

2d. *When the fraction formed in the preceding case, increased by the exponent of the parenthesis, is a whole number.*

These two rules are called the **conditions of integrability** of a binomial differential, and when either of them is fulfilled, the integration may be effected according to the general method above indicated.

We give an example under each case by way of illustration.

1. Integrate $x^3(1 + x^2)^{\frac{1}{2}}\, dx$.

Here, $\dfrac{m+1}{n} = \dfrac{4}{2} = 2$, a whole number, and therefore this comes under the first case.

Let $\qquad 1 + x^2 = z^2.$

Then
$$x^2 = z^2 - 1,$$
$$x^4 = (z^2 - 1)^2,$$
$$x^3\, dx = (z^2 - 1)\, z\, dz \qquad (1);$$

also
$$(1 + x^2)^{\frac{1}{2}} = z \qquad (2).$$

Multiplying (1) and (2) together, we have

$$x^3(1 + x^2)^{\frac{1}{2}}\, dx = (z^2 - 1)\, z^2 dz = z^4\, dz - z^2\, dz.$$

$$\therefore \int x^3 (1+x^2)^{\frac{1}{2}} dx = \frac{1}{5} z^5 - \frac{1}{3} z^3 + C,$$

$$= \frac{1}{5} (1+x^2)^{\frac{5}{2}} - \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C.$$

2. Integrate $x^{-4}(1+x^2)^{-\frac{1}{2}} dx$.

Here, $\frac{m+1}{n} + p = \frac{-4+1}{2} - \frac{1}{2} = -2$, a whole number, and therefore the second method is applicable.

Put $x^{-4}(1+x^2)^{-\frac{1}{2}} = x^{-5}(x^{-2}+1)^{-\frac{1}{2}}$; and let

$$x^{-2}+1 = z^2.$$

Then $\qquad x^{-3}\,dx = -z\,dz \qquad (1),$

$$x^{-2} = z^2 - 1 \qquad (2);$$

also $\qquad (x^{-2}+1)^{-\frac{1}{2}} = z^{-1} \qquad (3).$

Multiplying (1), (2), and (3), together, we have

$$x^{-5}(x^{-2}+1)^{-\frac{1}{2}} dx = -(z^2-1)\,dz.$$

$$\therefore \int = -\int (z^2-1)\,dz = -\left(\frac{z^3}{3} - z\right) = -z\left(\frac{z^2}{3} - 1\right)$$

$$= -\frac{\sqrt{1+x^2}}{x}\left(\frac{1+x^2}{3x^2} - 1\right) = \frac{(2x^2-1)}{3x^3}\sqrt{1+x^2} + C.$$

Formulas of Reduction.

25. Whenever a binomial differential can be rationalized, it may be integrated as in the last article; but usually the methods there set forth give rise to such complicated results that it is, as a general rule, more elegant to integrate by the process of systematic reduction, the formulas for which we propose now to establish.

26. Problem.—*To integrate* $x^m(a+bx^n)^p dx$ *by reducing the exponent* m *of the variable without the parenthesis.*

Let $$\int x^m (a+bx^n)^p dx = \int u dv = uv - \int v du;$$

and assume $u = x^{m-n+1}$; $dv = x^{n-1}(a+bx^n)^p dx$.

Then $$du = (m-n+1)x^{m-n} dx;$$

$$v = \frac{1}{nb(p+1)}(a+bx^n)^{p+1}.$$

$$\therefore \int x^m (a+bx^n)^p dx$$

$$= \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1} dx \quad (1).$$

This formula diminishes the exponent m to $m-n$, but increases p, which is generally objectionable. The last term of (1) must therefore be converted into an expression in which the exponent p shall not be increased.

Now we have

$$(a+bx^n)^{p+1} = (a+bx^n)(a+bx^n)^p = a(a+bx^n)^p + bx^n(a+bx^n)^p.$$

$$\therefore \; x^{m-n}(a+bx^n)^{p+1} = ax^{m-n}(a+bx^n)^p + bx^m(a+bx^n)^p.$$

Substituting in (1), we have

$$\int x^m (a+bx^n)^p dx$$

$$= \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{a(m-n+1)}{nb(p+1)}\int x^{m-n}(a+bx^n)^p dx$$

$$- \frac{m-n+1}{n(p+1)}\int x^m(a+bx^n)^p dx,$$

from which we obtain, by transposition and reduction,

$$\int x^m (a+bx^n)^p dx$$

$$= \frac{x^{m-n+1}(a+bx^n)^{p+1} - a(m-n+1)\int x^{m-n}(a+bx^n)^p dx}{b(np+m+1)} \quad [A].$$

This formula reduces the exponent of x^m to $m-n$. It may be used whenever m is positive and greater than n, and its continued application will finally reduce the expression to be integrated to a form which can be more readily manipulated than the original.

27. Problem.—*To obtain a formula for increasing* m;—to be used when m is negative.

We have from [A], by clearing of fractions, transposition and reduction,

$$\int x^{m-n}(a+bx^n)^p\,dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}-b(np+m+1)\int x^m(a+bx^n)^p\,dx}{a(m-n+1)} \quad . \text{ (2).}$$

If in this formula we write $-m$ for $m-n$, and $-m+n$ for m, we shall have

$$\int x^{-m}(a+bx^n)^p\,dx=-\frac{x^{-m+1}(a+bx^n)^{p+1}}{a(m-1)}$$

$$-\frac{b(m-np-n-1)}{a(m-1)}\int x^{-m+n}(a+bx^n)^p\,dx \quad [B].$$

28. Formulas [A] and [B] are to be used whenever the value of p is the same as that of the fractional exponent in some one of the fundamental forms; but as in a given example p may be any fraction whatever, it is necessary to establish formulas for changing the value of p also.

29. Problem.—*To integrate* $x^m(a+bx^n)^p\,dx$ *by diminishing the value of* p.

Let $\qquad (a+bx^n)^p=u; \quad x^m\,dx=dv.$

Then $\quad du=nbp(a+bx^n)^{p-1}x^{n-1}dx; \quad v=\dfrac{x^{m+1}}{m+1};$

and substituting in the formula for integration by parts, we have

$$\int x^m (a + bx^n)^p \, dx$$

$$= \frac{x^{m+1}(a + bx^n)^p}{m + 1} - \frac{nbp}{m + 1}\int x^{m+n}(a + bx^n)^{p-1} \, dx \quad (3).$$

Now, applying [A] to the last term of this equation, substituting $m + n$ for m, and $p - 1$ for p, we have

$$\int x^{m+n} (a + bx^n)^{p-1} \, dx$$

$$= \frac{x^{m+1} (a + bx^n)^p - a(m + 1)\int x^m (a + bx^n)^{p-1} \, dx}{b(np + m + 1)},$$

and by substitution in (3) we have

$$\int x^m (a + bx^n)^p \, dx$$

$$= \frac{x^{m+1} (a + bx^n)^p + anp\int x^m (a + bx^n)^{p-1} \, dx}{m + np + 1} \quad [C].$$

30. Problem.—*To obtain a formula for increasing* p;—to be used when p is negative.

We have from [C], by transposition and reduction,

$$\int x^m(a + bx^m)^{p-1} \, dx$$

$$= \frac{-x^{m+1}(a + bx^n)^p + (m + np + 1)\int x^m (a + bx^n)^p \, dx}{anp} \quad (4).$$

If in (4) we write $-p$ for $p - 1$, or $-p + 1$ for p, we shall have the required formula,

$$\int x^m (a + bx^n)^{-p} \, dx$$

$$= \frac{x^{m+1}(a+bx^n)^{-p+1} - (m - np + n + 1)\int x^m(a + bx^n)^{-p+1} \, dx}{an(p - 1)} \quad [D].$$

31. EXAMPLES.

1. Integrate $\dfrac{x^m\,dx}{\sqrt{1-x^2}} = x^m(1-x^2)^{-\frac{1}{2}}\,dx.$

Applying formula [A], making $m=m$, $a=1$, $b=-1$, $n=2$, $p=-\dfrac{1}{2}$, we have

$$\int\frac{x^m\,dx}{\sqrt{1-x^2}} = \frac{-x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int\frac{x^{m-2}\,dx}{\sqrt{1-x^2}} \quad (a).$$

If m be even, the continued application of this formula will reduce the integral to the form

$$\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C;$$

and if m be odd, the final expression to be integrated will be

$$\int\frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C.$$

1*st.* When m is even. Put $m=0$, 2, 4, 6, etc., in succession. Then, from (a), we shall have

$$\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C.$$

$$\int\frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}};$$

$$\int\frac{x^4\,dx}{\sqrt{1-x^2}} = \frac{-x^3\sqrt{1-x^2}}{4} + \frac{3}{4}\int\frac{x^2\,dx}{\sqrt{1-x^2}};$$

$$\int\frac{x^6\,dx}{\sqrt{1-x^2}} = \frac{-x^5\sqrt{1-x^2}}{6} + \frac{5}{6}\int\frac{x^4\,dx}{\sqrt{1-x^2}}.$$

. . . .

Hence, by reduction,

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\,x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C;$$

$$\int \frac{x^4\,dx}{\sqrt{1-x^2}} = -\left\{\frac{x^3}{4} + \frac{1}{2}\,\frac{3}{4}\,x\right\}\sqrt{1-x^2}$$
$$+\frac{1}{2}\,\frac{3}{4}\sin^{-1}x + C;$$

$$\int \frac{x^6\,dx}{\sqrt{1-x^2}} = -\left\{\frac{1}{6}\,x^5 + \frac{1}{4}\,\frac{5}{6}\,x^3 + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,x\right\}\sqrt{1-x^2}$$
$$+\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\sin^{-1}x + C;$$

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$$\int \frac{x^m\,dx}{\sqrt{1-x^2}} = -\left\{\frac{1}{m}\,x^{m-1} + \frac{1}{m-2}\,\frac{m-1}{m}\,x^{m-3}\right.$$
$$+\frac{1}{m-4}\,\frac{m-3}{m-2}\,\frac{m-1}{m}\,x^{m-5}$$
$$\left. + \cdots + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{m-1}{m}\,x\right\}\sqrt{1-x^2}$$
$$+\left\{\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{m-3}{m-2}\,\frac{m-1}{m}\right\}\sin^{-1}x + C.$$

If we take the value of this integral between the limits $x=0$ and $x=1$ [Art. 5], then, since for $x=0$, $\int = 0$, and for $x=1$, $\sin^{-1}x = \frac{\pi}{2}$, we shall have

$$\int_0^1 \frac{x^m\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{m-3}{m-2}\,\frac{m-1}{m}\,\frac{\pi}{2}.$$

2*d*. If m be an odd number, we shall have

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C;$$

$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = -\frac{x^2\sqrt{1-x^2}}{3} + \frac{2}{3}\int \frac{x\,dx}{\sqrt{1-x^2}};$$

$$\int \frac{x^5\,dx}{\sqrt{1-x^2}} = -\frac{x^4}{5}\sqrt{1-x^2} + \frac{4}{5}\int \frac{x^3\,dx}{\sqrt{1-x^2}};$$

$$\int \frac{x^7\,dx}{\sqrt{1-x^2}} = -\frac{x^6\sqrt{1-x^2}}{7} + \frac{6}{7}\int \frac{x^5\,dx}{\sqrt{1-x^2}};$$

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Hence, by reduction,

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C;$$

$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = -\left\{\frac{1}{3}x^2 + \frac{1}{1}\,\frac{2}{3}\right\}\sqrt{1-x^2} + C;$$

$$\int \frac{x^5\,dx}{\sqrt{1-x^2}} = -\left\{\frac{1}{5}x^4 + \frac{1}{3}\,\frac{4}{5}x^2 + \frac{1}{1}\,\frac{2}{3}\,\frac{4}{5}\right\}\sqrt{1-x^2} + C;$$

$$\int \frac{x^7\,dx}{\sqrt{1-x^2}} = -\left\{\frac{1}{7}x^6 + \frac{1}{5}\,\frac{6}{7}x^4 + \frac{1}{3}\,\frac{4}{5}\,\frac{6}{7}x^2 + \frac{1}{1}\,\frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\right\}\sqrt{1-x^2} + C;$$

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$$\int \frac{x^m\, dx}{\sqrt{1-x^2}} =$$

$$-\left\{\begin{array}{l} \frac{1}{m} x^{m-1} + \frac{1}{m-2}\,\frac{m-1}{m} x^{m-3} \\ \quad + \frac{1}{m-4}\,\frac{m-3}{m-2}\,\frac{m-1}{m} x^{m-5} \\ \quad + \cdots + \frac{1}{1}\,\frac{2}{3}\,\frac{4}{5} \cdots \frac{m-1}{m} \end{array}\right\} \sqrt{1-x^2} + C.$$

If the integral be taken between the limits $x = 0$ and $x = 1$, we shall have

$$\int_0^1 \frac{x^m\, dx}{\sqrt{1-x^2}} = \frac{1}{1}\,\frac{2}{3}\,\frac{4}{5}\,\frac{6}{7} \cdots \frac{m-1}{m}.$$

3*d*. If, finally, m be taken equal to *infinity*, the values of the definite integrals in the last two cases will be equal to each other, and we shall have

$$\frac{\pi}{2}\,\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\frac{7}{8} \cdots = \frac{1}{1}\,\frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\,\frac{8}{9} \cdots;$$

whence $$\frac{\pi}{2} = \frac{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8 \ldots}{1.\,3.\,3.\,5.\,5.\,7.\,7.\,9 \cdots},$$

which is **Wallis's theorem** for finding the ratio of the circumference of a circle to its diameter.

2. Integrate $\dfrac{dx}{x^m\sqrt{1-x^2}} = x^{-m}(1-x^2)^{-\frac{1}{2}}\, dx.$

This example will be integrated by formula [*B*].

We have

$$\int = -\frac{x^{-m+1}\sqrt{1-x^2}}{m-1} + \frac{m-2}{m-1}\int x^{-m+2}(1-x^2)^{-\frac{1}{2}}\, dx.$$

Proceeding as in the last example, we shall reduce the integration to that of $(1-x^2)^{-\frac{1}{2}}\, dx = \sin^{-1} x + C$, if m be even, and to that of $x^{-1}(1-x^2)^{-\frac{1}{2}}\, dx$, if m be odd.

To obtain the integral of this expression, assume

$$\sqrt{1-x^2}=1+xz;\ \text{whence}\ x=-2z-xz^2;$$

$$\frac{dx}{1+xz}=\frac{dx}{\sqrt{1-x^2}}=\frac{-2\,dz}{1+z^2};\quad \frac{dx}{x\sqrt{1-x^2}}=\frac{dz}{z}.$$

$$\therefore \int x^{-1}(1-x^2)^{-\frac{1}{2}}dx=\int\frac{dz}{z}=\log z+C$$

$$=\log\left\{\frac{\sqrt{1-x^2}-1}{x}\right\}+C.$$

3. Integrate $\dfrac{x^m\,dx}{\sqrt{2\,ax-x^2}}=x^m(2\,ax-x^2)^{-\frac{1}{2}}dx$

$$=x^{m-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}\,dx.$$

Formula [A] applied to the last form of this expression gives

$$\int\frac{x^m\,dx}{\sqrt{2\,ax-x^2}}=-\frac{x^{m-1}}{m}\sqrt{2\,ax-x^2}$$

$$+\frac{a}{m}(2\,m-1)\int\frac{x^{m-1}\,dx}{\sqrt{2\,ax-x^2}}.$$

Making $m=1, 2, 3, 4, 5$, etc., in succession, we have

$$\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}=-\sqrt{2\,ax-x^2}+a\int\frac{dx}{\sqrt{2\,ax-x^2}};$$

$$\int\frac{x^2\,dx}{\sqrt{2\,ax-x^2}}=-\frac{x}{2}\sqrt{2\,ax-x^2}+\frac{3}{2}\,a\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}$$

$$=-\left(\frac{x}{2}+\frac{3}{2}\,a\right)\sqrt{2\,ax-x^2}+\frac{3}{2}\,a^2\int\frac{dx}{\sqrt{2\,ax-x^2}};$$

$$\int \frac{x^3\,dx}{\sqrt{2ax-x^2}} = -\frac{x^2}{3}\sqrt{2ax-x^2} + \frac{5}{3}a\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}$$

$$= -\left(\frac{x^2}{3} + \frac{5}{3}\,\frac{x}{2}a + \frac{5}{3}\,\frac{3}{2}a^2\right)\sqrt{2ax-x^2}$$

$$+ \frac{5}{3}\,\frac{3}{2}a^3\int\frac{dx}{\sqrt{2ax-x^2}};$$

$$\int \frac{x^4\,dx}{\sqrt{2ax-x^2}} = -\frac{x^3}{4}\sqrt{2ax-x^2} + \frac{7}{4}a\int\frac{x^3\,dx}{\sqrt{2ax-x^2}}$$

$$= -\left(\frac{x^3}{4} + \frac{7}{4}\,\frac{x^2}{3}a + \frac{7}{4}\,\frac{5}{3}\,\frac{x}{2}a^2 + \frac{7}{4}\,\frac{5}{3}\,\frac{3}{2}a^3\right)\sqrt{2ax-x^2}$$

$$+ \frac{7}{4}\,\frac{5}{3}\,\frac{3}{2}a^4\int\frac{dx}{\sqrt{2ax-x^2}};$$

$$\int \frac{x^5\,dx}{\sqrt{2ax-x^2}} = -\frac{x^4}{5}\sqrt{2ax-x^2} + \frac{9}{5}a\int\frac{x^4\,dx}{\sqrt{2ax-x^2}}$$

$$= -\left(\frac{x^4}{5} + \frac{9}{5}\,\frac{x^3}{4}a + \frac{9}{5}\,\frac{7}{4}\,\frac{x^2}{3}a^2 + \frac{9}{5}\,\frac{7}{4}\,\frac{5}{3}\,\frac{x}{2}a^3\right.$$

$$\left. + \frac{9}{5}\,\frac{7}{4}\,\frac{5}{3}\,\frac{3}{2}a^4\right)\sqrt{2ax-x^2}$$

$$+ \frac{9}{5}\,\frac{7}{4}\,\frac{5}{3}\,\frac{3}{2}a^5\int\frac{dx}{\sqrt{2ax-x^2}};$$

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$$\int \frac{x^m\,dx}{\sqrt{2ax-x^2}} =$$

$$-\left\{\begin{matrix}\dfrac{x^{m-1}}{m} + \dfrac{2m-1}{m}\,\dfrac{x^{m-2}}{m-1}a + \dfrac{2m-1}{m}\,\dfrac{2m-3}{m-1}\,\dfrac{x^{m-3}}{m-2}a^2 \\ + \cdots + \dfrac{2m-1}{m}\,\dfrac{2m-3}{m-1}\,\dfrac{2m-5}{m-2}\cdots\dfrac{3}{2}\,\dfrac{1}{1}a^{m-1}\end{matrix}\right\}\sqrt{2ax-x^2}$$

$$+ \frac{2m-1}{m}\,\frac{2m-3}{m-1}\,\frac{2m-5}{m-2}\cdots\frac{3}{2}\,\frac{1}{1}a^m \operatorname{versin}^{-1}\frac{x}{a} + C.$$

If we take the integral between the limits $x=0$ and $x=2a$, we shall have $\text{versin}^{-1}\frac{2a}{a}=\pi$, and therefore,

$$\int_0^{2a}\frac{x^m\,dx}{\sqrt{2ax-x^2}}=\pi a^m\frac{1.\,3.\,\ldots\,2m-5.\,2m-3.\,2m-1}{1.\,2.\,\ldots\,\;m-2.\;\;m-1.\;\;m}.$$

4. Integrate $(1-2x^2)^{\frac{3}{2}}\,dx$.

Let $\quad m=0,\quad a=1,\quad b=-2,\quad n=2,\quad p=\frac{3}{2}$.

Then we shall have, by applying formula $[C]$,

$$\int=\frac{x}{4}(1-2x^2)^{\frac{3}{2}}+\frac{3}{4}\int(1-2x^2)^{\frac{1}{2}}\,dx.$$

Applying formula $[C]$ to the last term, we have

$$\frac{3}{4}\int(1-2x^2)^{\frac{1}{2}}\,dx=\frac{3}{4}\left\{\frac{x}{2}(1-2x^2)^{\frac{1}{2}}+\frac{1}{2}\int\frac{dx}{\sqrt{1-2x^2}}\right\}.$$

$$\therefore\ \int(1-2x^2)^{\frac{3}{2}}\,dx=\frac{x}{4}(1-2x^2)^{\frac{3}{2}}$$

$$+\frac{3}{8}x(1-2x^2)^{\frac{1}{2}}+\frac{3}{8\sqrt{2}}\int\frac{dx\sqrt{2}}{\sqrt{1-2x^2}}$$

$$=\frac{x}{4}(1-2x^2)^{\frac{3}{2}}+\frac{3}{8}x(1-2x^2)^{\frac{1}{2}}$$

$$+\frac{3}{8\sqrt{2}}\sin^{-1}x\sqrt{2}+C.$$

5. Integrate $\dfrac{dx}{x(a+bx)^{\frac{3}{2}}}=x^{-1}(a+bx)^{-\frac{3}{2}}\,dx$.

Let $\quad m=-1,\quad n=1,\quad -p=-\frac{3}{2}$.

Then we shall have, from formula [D],

$$\int = -\frac{2}{a(a+bx)^{\frac{1}{2}}} + \frac{1}{a}\int x^{-1}(a+bx)^{-\frac{1}{2}}\,dx.$$

To integrate the last term, assume $a + bx = z^2$;

then $\quad dx = \frac{2z\,dz}{b}; \quad x^{-1} = \frac{b}{z^2 - a}; \quad (a+bx)^{-\frac{1}{2}} = z^{-1}.$

$$\therefore \int x^{-1}(a+bx)^{-\frac{1}{2}}\,dx = \int \frac{2\,dz}{z^2 - a} = \frac{1}{\sqrt{a}} \log\left\{\frac{z - \sqrt{a}}{z + \sqrt{a}}\right\}$$

$$= \frac{1}{\sqrt{a}} \log\left\{\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}}\right\} + C.$$

$$\therefore \int \frac{dx}{x(a+bx)^{\frac{3}{2}}} = \frac{2}{a(a+bx)^{\frac{1}{2}}}$$

$$+ \frac{1}{a\sqrt{a}} \log\left\{\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}}\right\} + C.$$

6. Integrate $\dfrac{dx}{x(1+2x)^{\frac{5}{2}}}$.

$$\text{Ans. } \frac{2+3x}{3}\,\frac{4}{(1+2x)^{\frac{3}{2}}} + \log\left\{\frac{\sqrt{1+2x} - 1}{\sqrt{1+2x} + 1}\right\} + C.$$

7. Integrate $\dfrac{x\sqrt{x}\,dx}{1+x^2}$.

$$\text{Ans. } 2\sqrt{x} + \frac{1}{\sqrt{2}}\left\{\log\left(\frac{x+1+\sqrt{2x}}{\sqrt{1+x^2}}\right)\right.$$

$$\left. - \operatorname{tang}^{-1}\frac{\sqrt{2x}}{1-x}\right\} + C.$$

The last two examples may be integrated without the use of the reduction formulas.

CHAPTER V.

LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS.

32. There have been no general methods yet established for integrating, in finite terms, all possible forms in which these functions may be presented; but there are many cases in which the integration can be effected, either by reducing the given transcendental expressions to equivalent algebraic forms, or by reducing them to other expressions of the same kind, but of simpler form.

33. A large number of expressions coming under the general form $Pz^n\,dx$, in which z is transcendental, may be integrated in the following manner:

$$\text{Let } \int P\,dx = Q; \int Q\frac{dz}{dx}\,dx = R; \int R\frac{dz}{dx}dx = S; \text{ etc.}$$

Then the formula for integration by parts will give

$$\int Pz^n\,dx = Qz^n - n\int Qz^{n-1}\frac{dz}{dx}\,dx;$$

$$\int Qz^{n-1}\frac{dz}{dx}\,dx = Rz^{n-1} - (n-1)\int Rz^{n-2}\frac{dz}{dx}\,dx;$$

$$\int Rz^{n-2}\frac{dz}{dx}\,dx = Sz^{n-2} - (n-2)\int Sz^{n-3}\frac{dz}{dx}\,dx.$$

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Hence, by substitution,

$$\int Pz^n\,dx = Qz^n - nRz^{n-1} + n(n-1)Sz^{n-2}$$
$$- n(n-1)(n-2)\,Tz^{n-3} + \ldots \quad [E].$$

APPLICATIONS OF FORMULA [E].

1*st.* Let $P=1$, and $z=\log x$.

Then $Q=\int dx=x$; $R=\int x\,\frac{dx}{x}\,\frac{dx}{dx}=x$; $S=x$, etc.

$$\therefore \int \log^n x\,dx = x\{\log^n x - n\log^{n-1}x + n(n-1)\log^{n-2}x$$
$$\ldots \pm n(n-1)\ldots 3.2.1\}+C \qquad [F].$$

2*d.* Let $P=1$, and $z=\sin^{-1}x$.

Then we shall find

$$Q=x;\quad R=-\sqrt{1-x^2};\quad S=-x;\ \text{etc.}$$

$$\therefore \int(\sin^{-1}x)^n\,dx = x(\sin^{-1}x)^n + n\sqrt{1-x^2}(\sin^{-1}x)^{n-1}$$
$$-n(n-1)(x)(\sin^{-1}x)^{n-2}$$
$$-n(n-1)(n-2)\sqrt{1-x^2}(\sin^{-1}x)^{n-3}+\ldots \quad [G].$$

3*d.* Let $P=x^{m-1}$, and $z=\log x$.

Then $$Q=\int x^{m-1}\,dx=\frac{1}{m}x^m;$$

$$R=\frac{1}{m}\int x^{m-1}\,dx=\frac{1}{m^2}x^m,\ \text{etc.}$$

$$\therefore \int x^{m-1}\log^n x\,dx$$

$$=\frac{x^m}{m}\left\{\begin{matrix}\log^n x-\dfrac{n}{m}\log^{n-1}x+\dfrac{n(n-1)}{m^2}\log^{n-2}x\\ +\cdots\pm\dfrac{n(n-1)(n-2)\ldots 3.2.1}{m^n}\end{matrix}\right\}+C \qquad [H].$$

4th. If in the first formula [F] we put $\log x = z$, whence $x = e^z$, $dx = e^z\,dz$, we shall have

$$\int z^n e^z\,dz = e^z\{z^n - nz^{n-1} + n(n-1)z^{n-2} \dots$$
$$\pm n(n-1)(n-2) \dots 3.2.1\} + C \quad [I].$$

5th. Making the same substitution in [H], we have

$$x^{m-1}\,dx = e^{(m-1)z} e^z\,dz = e^{mz}\,dz.$$

$$\therefore \int z^n e^{mz}\,dz = \frac{e^{mz}}{m}\left\{z^n - \frac{n}{m}z^{n-1} + \frac{n(n-1)}{m^2}z^{n-2}\right.$$
$$\left. - \cdots \pm \frac{n(n-1)\dots 3.2.1}{m^n}\right\} + C \quad [K].$$

6th. In formula [G] we have

$$z = \sin^{-1} x; \quad \text{whence } x = \sin z,$$

$$dx = \cos z\,dz, \text{ and } \sqrt{1-x^2} = \cos z.$$

Substituting these values in [G], we have

$$\int z^n \cos z\,dz = \sin z\{z^n - n(n-1)z^{n-2} + \cdots\}$$
$$+ \cos z\{nz^{n-1} - n(n-1)(n-2)z^{n-3} + \dots\} + C \quad [L].$$

7th. If in formula [F] we make $x = a^z$, whence

$$dx = a^z \log a\,dz, \quad \log x = z \log a,$$

we shall have, after dividing through by $\log^{n+1} a$,

$$\int a^z z^n\,dz = \frac{a^z}{\log a}\left\{z^n - \frac{nz^{n-1}}{\log a} + \frac{n(n-1)}{\log^2 a}z^{n-2}\right.$$
$$\left. - \cdots + \frac{n(n-1)\dots 3.2.1}{\log^n a}\right\} + C \quad [M].$$

This formula embraces [I] as a particular case.

34. When n is *negative*, the above method no longer gives converging results, and the integral must then be determined by other means.

The following examples will serve for illustration:

1st. $\int \frac{x^m\,dx}{\log^n x}$.

Put $$x^{m+1} = u, \quad \frac{dx}{x\log^n x} = dv; \quad \text{then}$$

$$du = (m+1)\,x^m\,dx, \text{ and } v = -\frac{1}{(n-1)\log^{n-1} x}.$$

Substituting in the formula for integration by parts, we have

$$\int \frac{x^m\,dx}{\log^n x} = -\frac{x^{m+1}}{(n-1)\log^{n-1} x} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{\log^{n-1} x};$$

$$\int \frac{x^m\,dx}{\log^{n-1} x} = -\frac{x^{m+1}}{(n-2)\log^{n-2} x} + \frac{m+1}{n-2}\int \frac{x^m\,dx}{\log^{n-2} x};$$

$$\int \frac{x^m\,dx}{\log^{n-2} x} = -\frac{x^{m+1}}{(n-3)\log^{n-3} x} + \frac{m+1}{n-3}\int \frac{x^m\,dx}{\log^{n-3} x};$$

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$$\therefore \int \frac{x^m\,dx}{\log^n x} = -\frac{x^{m+1}}{n-1}\left\{\frac{1}{\log^{n-1} x} + \frac{m+1}{n-2}\,\frac{1}{\log^{n-2} x}\right.$$
$$+ \frac{(m+1)^2}{(n-2)(n-3)}\,\frac{1}{\log^{n-3} x} + \cdots$$
$$\left. + \frac{(m+1)^{n-2}}{(n-2)(n-3)\ldots 3.\,2.\,1}\,\frac{1}{\log x}\right\}$$
$$+ \frac{(m+1)^{n-1}}{(n-1)(n-2)(n-3)\ldots 3.\,2.\,1}\int \frac{x^m\,dx}{\log x} \quad [N].$$

2d. $\int \frac{a^x\,dx}{x^n}$.

If in [N] we put $x = a^u$, whence $dx = a^u \log a\,du$, and $\log x = u \log a$, we shall have

$$\int \frac{a^{(m+1)u}\,du}{u^n \log^{n-1} a} = -\frac{a^{(m+1)u}}{n-1}\left\{\frac{1}{u^{n-1}\log^{n-1} a}\right.$$

$$\left. + \frac{m+1}{(n-2)u^{n-2}\log^{n-2} a} + \cdots\right\}$$

$$+ \frac{(m+1)^{n-1}}{(n-1)(n-2)(n-3)\ldots 3.\,2.\,1}\int \frac{a^{(m+1)u}\,du}{u} \qquad [N'].$$

Making $m = 0$, and writing x for u, this formula becomes, after multiplying through by $\log^{n-1} a$,

$$\int \frac{a^x\,dx}{x^n} = -\frac{a^x}{(n-1)x^{n-1}}\left\{1 + \frac{x\log a}{n-2}\right.$$

$$\left. + \frac{x^2\log^2 a}{(n-2)(n-3)} + \cdots + \frac{x^{n-2}\log^{n-2} a}{(n-2)(n-3)\ldots 3.\,2.\,1}\right\}$$

$$+ \frac{\log^{n-1} a}{(n-1)(n-2)\ldots 3.\,2.\,1}\int \frac{a^x\,dx}{x} \qquad [O].$$

Finally, making $a = e$, whence $\log a = \log e = 1$, we have

$$\int \frac{e^x\,dx}{x^n} = -\frac{e^x}{(n-1)x^{n-1}}\left\{1 + \frac{x}{n-2}\right.$$

$$\left. + \frac{x^2}{(n-2)(n-3)} + \cdots + \frac{x^{n-2}}{(n-2)(n-3)\ldots 3.\,2.\,1}\right\}$$

$$+ \frac{1}{(n-1)(n-2)\ldots 3.\,2.\,1}\int \frac{e^x\,dx}{x} \qquad [O'].$$

The values of the last terms of the last four formulas can not be found except by approximation.

35. Trigonometric or Circular Forms. — The principal form to be integrated is

$$\sin^m x \cos^n x \, dx,$$

and it presents four cases for consideration: 1*st*, to reduce m; 2*d*, to reduce n; 3*d*, to increase m; 4*th*, to increase n.

Case I.—*To reduce* m.

Let $\sin^{m-1} x = u$, $\cos^n x \sin x \, dx = dv$; then

$$du = (m-1) \sin^{m-2} x \cos x \, dx, \quad \text{and} \quad v = -\frac{\cos^{n+1} x}{n+1}.$$

Substituting these values in the formula for integration by parts, we obtain

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1}$$

$$+ \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x \, dx \quad (1).$$

Now, $\displaystyle\int \sin^{m-2} x \cos^{n+2} x \, dx = \int \sin^{m-2} x \cos^n x \cos^2 x \, dx$

$$= \int \sin^{m-2} x \cos^n x \, (1 - \sin^2 x) \, dx$$

$$= \int \sin^{m-2} x \cos^n x \, dx - \int \sin^m x \cos^n x \, dx.$$

Therefore, by substitution in (1), and reduction,

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

$$+ \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx \quad [P].$$

Case II.—*To reduce* n.

If in [P] we write $90° - x$ for x, we shall have

$$\int \sin^m(90°-x) \cos^n(90°-x)\,dx$$

$$= \frac{\sin^{m-1}(90°-x)\cos^{n+1}(90°-x)}{m+n}$$

$$+ \frac{m-1}{m+n}\int \sin^{m-2}(90°-x)\cos^n(90°-x)\,dx:$$

or, observing that $\sin(90°-x) = \cos x$, $\cos(90°-x) = \sin x$, and changing m to n, we have

$$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n}$$

$$+ \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx \quad [Q].$$

The continued application of these two formulas will reduce the exponents m and n as near as possible to *zero*, and if they are both whole numbers, the final term to be integrated will be presented under one of the following forms, viz:

$$\int dx, \int \cos x\,dx, \int \sin x\,dx, \int \sin x \cos x\,dx.$$

The only exception is when $m + n = 0$, which will receive attention hereafter.

Case III.—*To increase* m.

If in formula [P] we put $-m$ for m, we shall have

$$\int \sin^{-m} \cos^n x\,dx$$

$$= \frac{-\sin^{-m-1} x \cos^{n+1} x - (m+1)\int \sin^{-m-2} x \cos^n x\,dx}{-m+n}.$$

$$\therefore \int \sin^{-m-2} x \cos^n x\,dx$$

$$= \frac{-\sin^{-m-1} x \cos^{n+1} x + (m-n)\int \sin^{-m} \cos^n x\,dx}{m+1}.$$

If in this equation we write $m-2$ for m, it becomes

$$\int \sin^{-m} x \cos^n x \, dx$$

$$= \frac{-\sin^{-m+1} x \cos^{n+1} x + (m-n-2)\int \sin^{-m+2} x \cos^n x \, dx}{m-1} \qquad [R].$$

This formula is not applicable when $m=1$, but in that case we may, by diminishing n, reduce the final integral to

$$\int \frac{\cos x \, dx}{\sin x}, \text{ or } \int \frac{dx}{\sin x}.$$

Case IV.—*To increase* n.

If in formula [Q] we write $-n$ for n, we shall have

$$\int \sin^m x \cos^{-n} x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{-n-1} x - (n+1)\int \sin^m x \cos^{-n-2} x \, dx}{m-n}.$$

$$\therefore \int \sin^m x \cos^{-n-2} x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{-n-1} x - (m-n)\int \sin^m x \cos^{-n} x \, dx}{n+1}.$$

If in this equation we write $n-2$ for n, it becomes

$$\int \sin^m x \cos^{-n} x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{-n+1} x - (m-n+2)\int \sin^m x \cos^{-n+2} x \, dx}{n-1} \qquad [S].$$

This formula is not applicable when $n=1$, but in that case we may, by diminishing m, reduce the final term to be integrated to

$$\int \frac{\sin x}{\cos x} \, dx, \text{ or } \int \frac{dx}{\cos x}.$$

36. Problem.—*To integrate* $\sin^{-m} x \cos^{-n} x\, dx$.

If in formula [P] we make m and n negative, we have

$$\int \sin^{-m} x \cos^{-n} x\, dx$$

$$= \frac{\sin^{-m-1} x \cos^{-n+1} x + (m+1) \int \sin^{-m-2} x \cos^{-n} x\, dx}{m+n}.$$

$$\therefore \int \sin^{-m-2} x \cos^{-n} x\, dx$$

$$= \frac{-\sin^{-m-1} x \cos^{-n+1} x + (m+n) \int \sin^{-m} x \cos^{-n} x\, dx}{m+1}.$$

If in this equation we write $m-2$ for m, it becomes

$$\int \sin^{-m} x \cos^{-n} x\, dx$$

$$= \frac{-\sin^{-m+1} x \cos^{-n+1} x + (m+n-2) \int \sin^{-m+2} x \cos^{-n} x\, dx}{m-1} \quad [T].$$

In precisely the same manner we obtain from [Q]

$$\int \sin^{-m} x \cos^{-n} x\, dx$$

$$= \frac{\sin^{-m+1} x \cos^{-n+1} x + (m+n-2) \int \sin^{-m} x \cos^{-n+2} x\, dx}{n-1} \quad [U].$$

By the continued application of these two formulas we may reduce the exponents m and n as near to zero as possible, and the final expressions to be integrated will be of the forms

$$\int \frac{dx}{\sin x}, \quad \int \frac{dx}{\cos x}, \quad \int \frac{dx}{\sin x \cos x}.$$

37. The integration, under all possible conditions, of the form

$$\int \sin^m x \cos^n x\, dx,$$

in which m and n are whole numbers, may be made to depend, as we have seen, upon the forms

$$\int dx,\quad \int \cos x\, dx,\quad \int \sin x\, dx,\quad \int \sin x \cos x\, dx,$$

$$\int \frac{\sin x}{\cos x}\, dx,\quad \int \frac{\cos x}{\sin x}\, dx,\quad \int \frac{dx}{\sin x \cos x},\quad \int \frac{dx}{\sin x},\quad \int \frac{dx}{\cos x},$$

$$\int \frac{\sin^m x}{\cos^m x}\, dx,\quad \int \frac{\cos^m x}{\sin^m x}\, dx,$$

each of which expressions we shall now integrate.

$$\int dx = x + C;$$

$$\int \cos x\, dx = \sin x + C;\quad \int \sin x\, dx = -\cos x + C;$$

$$\int \sin x \cos x\, dx = \int \sin x\, d(\sin x) = \frac{1}{2} \sin^2 x + C;$$

$$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{d \cos x}{\cos x} = -\log \cos x = \log \sec x + C;$$

$$\int \frac{\cos x}{\sin x}\, dx = \log \sin x + C;\quad \int \frac{dx}{\sin x \cos x}$$

$$= \int \frac{\dfrac{dx}{\cos^2 x}}{\dfrac{\sin x}{\cos x}} = \int \frac{\sec^2 x\, dx}{\tang x} = \int \frac{d \tang x}{\tang x} = \log \tang x + C;$$

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\dfrac{dx}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \log \tang \frac{x}{2} + C;$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(90^\circ - x)} = -\int \frac{d(90^\circ - x)}{\sin(90^\circ - x)}$$

$$= -\log \tang\left(\frac{90^\circ - x}{2}\right) + C.$$

To integrate $\frac{\sin^m x}{\cos^m x} dx$. If in the formula

$$\int \sin^m x \cos^n x \, dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x + (m-1)\int \sin^{m-2} x \cos^{n+2} x \, dx}{n+1}$$

we write $-m$ for n, we shall have

$$\int \sin^m x \cos^{-m} x \, dx = \int \frac{\sin^m x}{\cos^m x} dx = \int \operatorname{tang}^m x \, dx$$

$$= \frac{\operatorname{tang}^{m-1} x}{m-1} - \int \operatorname{tang}^{m-2} x \, dx \quad [V].$$

The application of this formula will reduce the form to be integrated to $\int dx$ or $\int \operatorname{tang} x \, dx = \int \frac{\sin x}{\cos x} dx$, both of which are known.

If in formula $[V]$ we write $(90° - x)$ for x, we have

$$\int \frac{\cos^m x \, dx}{\sin^m x} = \int \cot^m x \, dx$$

$$= -\frac{\operatorname{tang}^{m-1}(90° - x)}{m-1} - \int \operatorname{tang}^{m-2}(90° - x) \, dx$$

$$= -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x \, dx \qquad [W].$$

This formula reduces the final term to be integrated to

$$\int dx \text{ or } \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx,$$

both of which are known.

NOTE.—If, in the expression $\sin^m x \cos^n x\, dx$, the exponents m and n be fractional, the integration may be effected by transforming the given expression into an equivalent algebraic form, and then applying some one or more of the formulas applicable to such cases.

For example, let $m = \frac{3}{2}$, $n = \frac{5}{3}$; then the expression to be integrated is

$$\sin^{\frac{3}{2}} x \cos^{\frac{5}{3}} x\, dx \qquad (1).$$

Assume $\sin x = u$.

Then $\cos x = (1 - u^2)^{\frac{1}{2}}$;

$$dx = \frac{du}{\sqrt{1 - u^2}} = (1 - u^2)^{-\frac{1}{2}} du;$$

and by substitution in (1) we have

$$\int \sin^{\frac{3}{2}} x \cos^{\frac{5}{3}} x\, dx = \int u^{\frac{3}{2}} (1 - u^2)^{\frac{5}{6}} (1 - u^2)^{-\frac{1}{2}} du$$

$$= \int u^{\frac{3}{2}} (1 - u^2)^{\frac{1}{3}} du.$$

Finding the integral of this in terms of u, and then replacing u by its value $\sin x$, we shall have the required integral in terms of $\sin x$.

38. In all of the preceding formulas the integration has been effected in terms of the *powers* of the trigonometrical functions, sine, cosine, etc.

The reduction of these formulas to numerical results, in practical operations, renders necessary an excessive amount of arithmetical computation. This labor may be obviated, in general, by converting the powers of sines, etc., into the sines and cosines of *multiple* arcs before the integration is performed.

For this purpose we employ the three well-known trigonometrical formulas [Ray's Surveying, Art. 96]:

$$\sin a \sin b = \frac{1}{2} \cos (a - b) - \frac{1}{2} \cos (a + b) \quad (1),$$

$$\sin a \cos b = \frac{1}{2} \sin (a - b) + \frac{1}{2} \sin (a + b) \quad (2),$$

$$\cos a \cos b = \frac{1}{2} \cos (a - b) + \frac{1}{2} \cos (a + b) \quad (3).$$

If, for example, it be required to integrate

$$\sin^3 x \cos^2 x \, dx,$$

we have $\sin^3 x \cos^2 x \, dx = \sin x (\sin x \cos x)^2 \, dx$

$$= \sin x \left(\frac{1}{2} \sin 2x \right)^2 dx \text{ [by (2)]}$$

$$= \frac{1}{4} \sin x \left(\frac{1 - \cos 4x}{2} \right) dx \text{ [by (1)]}$$

$$= \left(\frac{1}{8} \sin x - \frac{1}{8} \sin x \cos 4x \right) dx$$

$$= \frac{1}{8} \sin x \, dx + \frac{1}{16} \sin 3x \, dx$$

$$- \frac{1}{16} \sin 5x \, dx \text{ [by (2)]}$$

$$\therefore \int = \int \frac{1}{8} \sin x \, dx + \int \frac{1}{16} \sin 3x \, dx - \int \frac{1}{16} \sin 5x \, dx$$

$$= - \frac{1}{8} \cos x - \frac{1}{48} \cos 3x + \frac{1}{80} \cos 5x + C.$$

39. We shall close this chapter by integrating several expressions of frequent occurrence.

1*st*. $\int e^{ax} \cos bx \, dx$;

2*d*. $\int e^{ax} \sin bx \, dx$.

Putting $e^{ax}\,dx = dv$, and $u = \cos bx$, $\sin bx$, respectively, we have, from the formula for integration by parts,

$$\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx,$$

$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx.$$

From these two equations we obtain by elimination,

$$\int e^{ax}\cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}\,e^{ax} \qquad [X],$$

$$\int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}\,e^{ax} \qquad [Y].$$

3*d*. $\displaystyle\int\frac{dx}{a+b\cos^2 x + c\sin^2 x}$.

We have

$$\int = \int\frac{\dfrac{dx}{\cos^2 x}}{\dfrac{a}{\cos^2 x}+b+c\,\text{tang}^2 x} = \int\frac{\sec^2 x\,dx}{a\sec^2 x + b + c\,\text{tang}^2 x}$$

$$= \int\frac{d\,\text{tang}\,x}{a+b+(a+c)\,\text{tang}^2 x} = \frac{1}{a+b}\int\frac{d\,\text{tang}\,x}{1+\dfrac{a+c}{a+b}\,\text{tang}^2 x}$$

$$= \frac{1}{a+b}\sqrt{\frac{a+b}{a+c}}\int\frac{d\left\{\sqrt{\dfrac{a+c}{a+b}}\,\text{tang}\,x\right\}}{1+\left\{\sqrt{\dfrac{a+c}{a+b}}\,\text{tang}\,x\right\}^2}$$

$$= \frac{1}{\sqrt{(a+b)(a+c)}}\,\text{tang}^{-1}\left\{\sqrt{\frac{a+c}{a+b}}\,\text{tang}\,x\right\} + C.$$

4th. $\int \frac{dx}{a + b\cos x}$ when $a > b$.

Since $\cos x = \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$, we have

$$\int = \int \frac{dx}{a + b\cos^2\frac{1}{2}x - b\sin^2\frac{1}{2}x}$$

$$= 2\int \frac{d\frac{x}{2}}{a + b\cos^2\frac{1}{2}x - b\sin^2\frac{1}{2}x}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \operatorname{tang}^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\operatorname{tang}\frac{x}{2}\right\} + C.$$

5th. $\int \frac{dx}{a + b\cos x}$ when $a < b$.

We have

$$\int = \int \frac{dx}{a\cos^2\frac{1}{2}x + a\sin^2\frac{1}{2}x + b\cos^2\frac{1}{2}x - b\sin^2\frac{1}{2}x}$$

$$= 2\int \frac{d\frac{x}{2}}{(a+b)\cos^2\frac{1}{2}x - (b-a)\sin^2\frac{1}{2}x}$$

$$= 2\int \frac{\sec^2\frac{x}{2}\,d\frac{x}{2}}{(a+b) - (b-a)\operatorname{tang}^2\frac{1}{2}x}$$

$$= \frac{2}{a+b}\sqrt{\frac{a+b}{b-a}}\int \frac{d\left\{\sqrt{\frac{b-a}{a+b}}\operatorname{tang}\frac{1}{2}x\right\}}{1 - \left\{\sqrt{\frac{b-a}{a+b}}\operatorname{tang}\frac{1}{2}x\right\}^2}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left\{ \frac{1 + \sqrt{\frac{b-a}{a+b}} \operatorname{tang} \frac{1}{2} x}{1 - \sqrt{\frac{b-a}{a+b}} \operatorname{tang} \frac{1}{2} x} \right\} + C$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left\{ \frac{\sqrt{b+a} + \sqrt{b-a} \operatorname{tang} \frac{1}{2} x}{\sqrt{b+a} - \sqrt{b-a} \operatorname{tang} \frac{1}{2} x} \right\} + C.$$

6*th*. $\int \frac{dx}{(a + b \cos x)^m}$.

Assume $u = \int = \frac{A \sin x}{(a + b \cos x)^{m-1}} + \int \frac{B + C \cos x}{(a + b \cos x)^{m-1}} dx,$

in which A, B, and C are unknown coëfficients.

Differentiating with respect to x, we have

$$\frac{du}{dx} = \frac{1}{(a + b \cos x)^m}$$

$$= \frac{A \cos x(a + b \cos x)^{m-1} + Ab(m-1)(a + b \cos x)^{m-2} \sin^2 x}{(a + b \cos x)^{2m-2}}$$

$$+ \frac{B + C \cos x}{(a + b \cos x)^{m-1}}$$

$$= \frac{A \cos x (a + b \cos x) + Ab(m-1)(1 - \cos^2 x) + (B + C \cos x)(a + b \cos x)}{(a + b \cos x)^m}$$

$$\therefore\ 1 = A \cos x (a + b \cos x) + Ab (m-1)(1 - \cos^2 x)$$
$$+ (B + C \cos x)(a + b \cos x);$$

from which we find, by the method of indeterminate coëfficients,

$$A = \frac{b}{(m-1)(b^2 - a^2)}; \quad B = \frac{-a}{b^2 - a^2}; \quad C = (m-2)A.$$

$$\therefore\ u = \int \frac{dx}{(a + b\cos x)^m} = \frac{b\sin x}{(m-1)(b^2 - a^2)(a + b\cos x)^{m-1}}$$

$$- \frac{a}{b^2 - a^2}\int \frac{dx}{(a + b\cos x)^{m-1}}$$

$$+ \frac{(m-2)\,b}{(m-1)(b^2 - a^2)}\int \frac{\cos x\,dx}{(a + b\cos x)^{m-1}}$$

This operation reduces the given integral to three terms, one of which is similar to the given expression with the exponent of the denominator diminished by unity.

7th. $\displaystyle\int \frac{\cos x\,dx}{(a + b\cos x)^{m-1}}$.

By a process entirely similar to that employed in the last example we find

$$\int = -\frac{a\sin x}{(m-2)(b^2 - a^2)(a + b\cos x)^{m-2}}$$

$$+ \frac{b}{b^2 - a^2}\int \frac{dx}{(a + b\cos x)^{m-2}}$$

$$- \frac{(m-3)\,a}{(m-2)(b^2 - a^2)}\int \frac{\cos x\,dx}{(a + b\cos x)^{m-2}}.$$

The continued application of the formulas of the last two examples will reduce the integration of the expression

$$\int \frac{dx}{(a + b\cos x)^m}$$

to that of a series of similar terms, in each of which the exponent of the denominator is one less than in the preceding; the final terms to be integrated being of the forms

$$\int \frac{dx}{a + b\cos x},\quad \int \cos x\,dx,\ \text{ or } \int \frac{\cos x\,dx}{a + b\cos x},$$

each of which is easily integrated.

For the last of these forms we have

$$\int \frac{\cos x\,dx}{a+b\cos x}=\int \frac{a+b\cos x-a}{b(a+b\cos x)}\,dx$$

$$=\int \frac{dx}{b}-\int \frac{a\,dx}{b(a+b\cos x)}$$

$$=\frac{x}{b}-\frac{a}{b}\int \frac{dx}{a+b\cos x}.$$

40. EXAMPLES.

1. Integrate $\dfrac{1+x^2}{(1+x)^2}e^x\,dx$.

We have $\dfrac{1+x^2}{(1+x)^2}=1-\dfrac{2x}{(1+x)^2}$;

$$\therefore \int = \int e^x\,dx-2\int \frac{xe^x\,dx}{(1+x)^2}.$$

To integrate the second term, put $1+x=z$, $dx=dz$, $e^x=e^{z-1}$. Then

$$\int \frac{xe^x\,dx}{(1+x)^2}=\int \frac{(z-1)e^{z-1}\,dz}{(z)^2}=\frac{1}{e}\int \frac{e^z}{z}\,dz-\frac{1}{e}\int \frac{e^z\,dz}{z^2}.$$

$$\therefore \int = e^x-\frac{2}{e}\int \frac{e^z}{z}\,dz+\frac{2}{e}\int \frac{e^z\,dz}{z^2}$$

$$=e^x-\frac{2}{e}\left\{\int \frac{e^z}{z}\,dz-\int \frac{e^z\,dz}{z^2}\right\}$$

$$=e^x-\frac{2}{e}\left\{\int \frac{e^z\,dz}{z}+\frac{e^z}{z}-\int \frac{e^z\,dz}{z}\right\},\text{ by formula } [O],$$

$$=e^x-\frac{2}{e}\,\frac{e^z}{z}=e^x-\frac{2}{e}\,\frac{e^{1+x}}{1+x}$$

$$=e^x\left\{1-\frac{2}{1+x}\right\}=e^x\left\{\frac{x-1}{x+1}\right\}+C.$$

2. Integrate $\dfrac{x^3\,dx}{\log^4 x}$.

Applying formula [N], we have

$$\int = -\frac{x^4}{3}\left\{\frac{1}{\log^3 x}+\frac{4}{2}\,\frac{1}{\log^2 x}\right.$$
$$\left.+\frac{4^2}{2.\,1}\,\frac{1}{\log x}\right\}+\frac{4^3}{3.\,2.\,1}\int\frac{x^3\,dx}{\log x}.$$

To integrate the last term, assume $x^4 = z$, $\log x = \frac{1}{4}\log z$, $x^3\,dx = \frac{1}{4}\,dz$. Then $\displaystyle\int\frac{x^3\,dx}{\log x} = \int\frac{dz}{\log z}$.

Now assume $z = e^t$; $dz = e^t\,dt$, $t = \log z$; then

$$\int\frac{dz}{\log z} = \int\frac{e^t\,dt}{t} = \int\left\{1+t+\frac{t^2}{1.\,2}+\text{etc.}\right\}\frac{dt}{t}$$
$$= \log t + t + \frac{t^2}{4} + \text{etc.}$$
$$= \log(\log x^4) + \log x^4 + \frac{1}{4}\log^2 x^4 + \text{etc.}$$

$$\therefore \int\frac{x^3\,dx}{\log^4 x} = -\frac{x^4}{3}\left\{\frac{1}{\log^3 x}+\frac{4}{2}\,\frac{1}{\log^2 x}+\frac{4^2}{2.\,1}\,\frac{1}{\log x}\right\}$$
$$+\frac{4^3}{3.\,2.\,1}\left\{\log(\log x^4)+\log x^4+\frac{1}{4}\log^2 x^4+\text{etc.}\right\}.$$

3. Integrate $\sin^5 x\,\cos^5 x\,dx$.

We have, by formula [P],

$$\int\sin^5 x\,\cos^5 x\,dx = -\frac{\sin^4 x\cos^6 x}{10}+\frac{4}{10}\int\sin^3 x\,\cos^5 x\,dx;$$

$$\int\sin^3 x\,\cos^5 x\,dx = -\frac{\sin^2 x\cos^6 x}{8}+\frac{2}{8}\int\sin x\,\cos^5 x\,dx;$$

and $\int \sin x \cos^5 x\,dx = -\int \cos^5 x d\cos x = -\frac{1}{6}\cos^6 x + C.$

$$\therefore \int = -\frac{\cos^6 x}{10}\left\{\sin^4 x + \frac{4}{8}\sin^2 x + \frac{1}{6}\right\} + C.$$

4. Integrate $\sin^6 x \cos^6 x\,dx$.

We have, from formula [P],

$$\int \sin^6 x \cos^6 x\,dx = -\frac{\sin^5 x \cos^7 x}{12} + \frac{5}{12}\int \sin^4 x \cos^6 x\,dx;$$

$$\int \sin^4 x \cos^6 x\,dx = -\frac{\sin^3 x \cos^7 x}{10} + \frac{3}{10}\int \sin^2 x \cos^6 x\,dx;$$

$$\int \sin^2 x \cos^6 x\,dx = -\frac{\sin x \cos^7 x}{8} + \frac{1}{8}\int \cos^6 x\,dx.$$

$$\therefore \int = -\frac{\cos^7 x}{12}\left\{\sin^5 x + \frac{5}{10}\sin^3 x\right.$$

$$\left. + \frac{3}{8}\,\frac{5}{10}\sin x\right\} + \frac{3.\ 5}{8.\ 10.\ 12}\int \cos^6 x\,dx.$$

Applying formula [Q] to the last term, we have

$$\int \cos^6 x\,dx = \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\int \cos^4 x\,dx;$$

$$\int \cos^4 x\,dx = \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,dx;$$

$$\int \cos^2 x\,dx = \frac{\sin x \cos x}{2} + \frac{1}{2}x + C.$$

$$\therefore \int \cos^6 x\,dx = \frac{\sin x}{6}\left\{\cos^5 x + \frac{5}{4}\cos^3 x\right.$$

$$\left. + \frac{3.\ 5}{2.\ 4}\cos x\right\} + \frac{3.\ 5}{2.\ 4.\ 6}x + C.$$

$$\therefore \int \sin^6 x \cos^6 x \, dx$$

$$= -\frac{\cos^7 x}{12}\left\{\sin^5 x + \frac{5}{10}\sin^3 x + \frac{3}{8}\,\frac{5}{10}\sin x\right\}$$

$$+ \frac{3.\,5 \sin x}{6.\,8.\,10.\,12}\left\{\cos^5 x + \frac{5}{4}\cos^3 x + \frac{3}{2}\,\frac{5}{4}\cos x\right\}$$

$$+ \frac{3^2.\,5^2}{2.\,4.\,6.\,8.\,10.\,12}\,x + C.$$

5. Integrate $\dfrac{dx}{\sin^4 x \cos^6 x}$.

This may be integrated by applying the necessary formulas, but we shall employ another process which is applicable in similar cases, and is especially advantageous when the exponents of $\sin x$ and $\cos x$ are even numbers.

Since $\sin^2 x + \cos^2 x = 1$, we have

$$\int \frac{dx}{\sin^4 x \cos^6 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x \cos^6 x}\,dx$$

$$= \int \frac{dx}{\sin^2 x \cos^6 x} + \int \frac{dx}{\sin^4 x \cos^4 x}$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^6 x}\,dx + \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x \cos^4 x}\,dx$$

$$= \int \frac{dx}{\cos^6 x} + \int \frac{dx}{\sin^2 x \cos^4 x} + \int \frac{dx}{\sin^2 x \cos^4 x} + \int \frac{dx}{\sin^4 x \cos^2 x}$$

$$= \int \frac{dx}{\cos^6 x} + 2\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^4 x}\,dx + \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x \cos^2 x}\,dx$$

$$= \int \frac{dx}{\cos^6 x} + 2\int \frac{dx}{\cos^4 x} + 2\int \frac{dx}{\sin^2 x \cos^2 x}$$

$$+ \int \frac{dx}{\sin^2 x \cos^2 x} + \int \frac{dx}{\sin^4 x}$$

$$=\int \frac{dx}{\cos^6 x} + 2\int \frac{dx}{\cos^4 x} + 3\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{dx}{\sin^4 x}$$

$$=\int \frac{dx}{\cos^6 x} + 2\int \frac{dx}{\cos^4 x} + 3\int \frac{dx}{\cos^2 x} + 3\int \frac{dx}{\sin^2 x} + \int \frac{dx}{\sin^4 x},$$

each term of which is integrable by the formulas.

6. Integrate $\text{tang}^5 x\, dx$.

Applying formula $[V]$, we have

$$\int \text{tang}^5 x\, dx = \frac{\text{tang}^4 x}{4} - \int \text{tang}^3 x\, dx;$$

$$\int \text{tang}^3 x\, dx = \frac{\text{tang}^2 x}{2} - \int \text{tang}\, x\, dx;$$

and $\displaystyle\int \text{tang}\, x\, dx = -\log \cos x + C.$

$$\therefore \int = \frac{\text{tang}^4 x}{4} - \frac{1}{2} \text{tang}^2 x - \log \cos x + C.$$

7. Integrate $\sin^5 x \cos^4 dx$ in terms of multiple arcs.

Referring to the formulas of Art. 38, we have

$$\sin^5 x \cos^4 x\, dx = \sin x (\sin x \cos x)^4 dx;$$

$$(\sin x \cos x)^4 = \frac{1}{16} \sin^4 2x;$$

$$\sin^4 2x = (\sin^2 2x)^2 = \left(\frac{1 - \cos 4x}{2}\right)^2$$

$$= \frac{1}{4} (1 - 2 \cos 4x + \cos^2 4x);$$

$$\cos^2 4x = \frac{1}{2} + \frac{1}{2} \cos 8x;$$

$\therefore \sin^5 x \cos^4 x$

$$= \frac{1}{64} \sin x \left(1 - 2 \cos 4x + \frac{1}{2} + \frac{1}{2} \cos 8x\right)$$

$$=\frac{1}{64}\sin x-\frac{1}{32}\sin x\cos 4x+\frac{1}{128}\sin x+\frac{1}{128}\sin x\cos 8x$$

$$=\frac{3}{128}\sin x-\frac{1}{32}\left(-\frac{1}{2}\sin 3x+\frac{1}{2}\sin 5x\right)$$

$$+\frac{1}{128}\left(-\frac{1}{2}\sin 7x+\frac{1}{2}\sin 9x\right)$$

$$=\frac{3}{128}\sin x+\frac{1}{64}\sin 3x-\frac{1}{64}\sin 5x$$

$$-\frac{1}{256}\sin 7x+\frac{1}{256}\sin 9x.$$

$$\therefore \int=-\frac{3}{128}\cos x-\frac{1}{3.64}\cos 3x+\frac{1}{5.64}\cos 5x$$

$$+\frac{1}{7.256}\cos 7x-\frac{1}{9.256}\cos 9x+C.$$

8. Integrate $\cos^6 x\,dx$.

We have

$$\cos^6 x=(\cos x\cos x)^3=\left(\frac{1+\cos 2x}{2}\right)^3$$

$$=\frac{1}{8}+\frac{3}{8}\cos 2x+\frac{3}{8}\cos^2 2x+\frac{1}{8}\cos^3 2x;$$

$$\cos^2 2x=\frac{1}{2}+\frac{1}{2}\cos 4x;$$

$$\cos^3 2x=\cos 2x\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)=\frac{1}{2}\cos 2x+\frac{1}{2}\cos 2x\cos 4x;$$

$$\cos 2x\cos 4x=\frac{1}{2}\cos 2x+\frac{1}{2}\cos 6x.$$

$$\therefore \cos^6 x=\frac{1}{8}+\frac{3}{8}\cos 2x+\frac{3}{16}+\frac{3}{16}\cos 4x$$

$$+\frac{1}{16}\cos 2x+\frac{1}{32}\cos 2x+\frac{1}{32}\cos 6x$$

$$=\frac{5}{16}+\frac{15}{32}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{32}\cos 6x.$$

$$\therefore \int=\frac{5}{16}x+\frac{15}{64}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{192}\sin 6x+C.$$

9. Integrate $\sin^2 x\cos^3 x\,dx$, as above.

10. Integrate $\sin^5 x\,dx$, as above.

11. As an additional application of the formula for integration by parts, we shall find the integral of

$$e^{ax}\sin^n x\,dx.$$

Let $\sin x\,dx=dv$, and $e^{ax}\sin^{n-1}x=u$;

then $v=-\cos x$, and $du=ae^{ax}\sin^{n-1}x\,dx$

$$+(n-1)e^{ax}\sin^{n-2}x\cos x\,dx.$$

$$\therefore \int=-e^{ax}\sin^{n-1}x\cos x+a\int e^{ax}\sin^{n-1}x\cos x\,dx$$

$$+(n-1)\int e^{ax}\sin^{n-2}x\cos^2 x\,dx \quad (1).$$

The last term in this equation may be put under the form

$$(n-1)\int e^{ax}\sin^{n-2}x\,dx-(n-1)\int e^{ax}\sin^n x\,dx \quad (2).$$

If in the second term we put $e^{ax}=u$, and $\sin^{n-1}x\cos x\,dx$ $=dv$, we shall have,

$$a\int e^{ax}\sin^{n-1}x\cos x\,dx=\frac{ae^{ax}\sin^n x}{n}-\frac{a^2}{n}\int e^{ax}\sin^n x\,dx \quad (3).$$

Substituting (2) and (3) in (1), we have

$$\int e^{ax}\sin^n x\,dx = -\,e^{ax}\sin^{n-1}x\cos x$$

$$+\frac{a}{n}e^{ax}\sin^n x - \frac{a^2}{n}\int e^{ax}\sin^n x\,dx$$

$$+(n-1)\int e^{ax}\sin^{n-2}x\,dx - (n-1)\int e^{ax}\sin^n x\,dx;$$

and, by transposition and reduction,

$$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x\,(a\sin x - n\cos x)}{a^2+n^2}$$

$$+\frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx \qquad [Z].$$

Similarly we shall find

$$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x)}{a^2+n^2}$$

$$+\frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx \qquad [Z'].$$

The continued application of these two formulas will reduce the final term to be integrated to forms which have been already considered.

Scholium.—There are many other general forms which admit of integration by the application of the formula

$$\int u\,dv = uv - \int v\,du.$$

Those which have been presented in the foregoing chapters are, however, among the most important, and of most frequent occurrence in the practical applications of the Calculus; and the careful study of these formulas should enable the student to apply the method to any not too complicated expression which may present itself for solution.

CHAPTER VI.

APPROXIMATE INTEGRATION. DEFINITE INTEGRALS. DIFFERENTIATION AND INTEGRATION UNDER THE SIGN $\int$.

41. When a differential expression can not be integrated in finite terms by any of the preceding methods, the operation may ofteu be effected by developing the given expression into a series, each term of which can be integrated separately; and this method may frequently be used conjointly with those established in the preceding chapters for the purpose of discovering the developed form of a known integral.

EXAMPLES.

1. To develop $\int F(x)\,dx$.

We have from Maclaurin's formula

$$F(x) = F(0) + x\,F'(0) + \frac{x^2}{1.2}F''(0) + \cdots \frac{x^m}{1.2\ldots m}F^m(0).$$

Multiplying each term of this equation by dx, and integrating, we have

$$\int F(x)\,dx = C + x\,F(0) + \frac{x^2}{1.2}F'(0) + \frac{x^3}{1.2.3}F''(0) + \cdots \frac{x^{m+1}}{1.2\ldots(m+1)}F^m(0),$$

which is the required development.

It is easily seen that this is the development, by Maclaurin's formula, of $\int F(x)\,dx$, C being the value of $\int F(x)\,dx$ when $x = 0$.

2. To develop $\int F(x)\,dx$.

Making $F(x) = u$, and $dx = dv$, we have, by substitution in the formula for integration by parts,

$$\int F(x)\,dx = x\,F(x) - \int x\,F'(x)\,dx;$$

$$\int x\,F'(x)\,dx = \frac{x^2}{1.\,2}\,F'(x) - \int \frac{x^2}{1.\,2}\,F''(x)\,dx;$$

$$\int \frac{x^2}{1.\,2}\,F''(x)\,dx = \frac{x^3}{1.\,2.\,3}\,F''(x) - \int \frac{x^3}{1.\,2.\,3}\,F'''(x)\,dx;$$

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Hence, by substitution and reduction,

$$\int F(x)\,dx = x\,F(x) - \frac{x^2}{1.\,2}\,F'(x) + \frac{x^3}{1.\,2.\,3}\,F''(x) - \frac{x^4}{1.\,2.\,3.\,4}\,F'''(x) + \text{etc.}$$

This formula is known as **Bernoulli's** series.

3. To develop $\int \frac{dx}{\sqrt{1-x^2}}$.

Expanding by the binomial formula, we obtain

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\,\frac{3}{4}x^4 + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}x^6 + \text{etc.}$$

$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = \int\left(dx + \frac{1}{2}\,x^2\,dx + \frac{1}{2}\,\frac{3}{4}\,x^4\,dx + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,x^6\,dx, \text{ etc.}\right)$$

$$= x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1}{2}\,\frac{3}{4}\,\frac{x^5}{5} + \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\frac{x^7}{7} + \text{etc.} + C'.$$

But $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

$$\therefore \sin^{-1} x + C = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \text{etc.} + C'.$$

If we suppose the integral to begin when $x = 0$, we shall have $C = C'$, and therefore

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \text{etc.}$$

This development will evidently not be applicable to arcs greater than one-fourth of the circumference.

Corollary.—If $x = 1$, then $\sin^{-1} x = \frac{\pi}{2}$, and

$$\therefore \frac{\pi}{2} = 1 + \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{3}{4}\frac{1}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7} + \text{etc.}$$

4. To develop $\int \frac{dx}{\sqrt{2ax - x^2}\sqrt{1 - bx}}$.

Expanding by the binomial formula, we obtain

$$(1 - bx)^{-\frac{1}{2}} = 1 + \frac{1}{2} bx + \frac{1}{2}\frac{3}{4} b^2 x^2 + \frac{1}{2}\frac{3}{4}\frac{5}{6} b^3 x^3 + \text{etc.}$$

$$\therefore \int \frac{dx}{\sqrt{2ax - x^2}\sqrt{1 - bx}} = \int \frac{dx}{\sqrt{2ax - x^2}} + \frac{1}{2} b \int \frac{x\,dx}{\sqrt{2ax - x^2}} + \frac{1}{2}\frac{3}{4} b^2 \int \frac{x^2\,dx}{\sqrt{2ax - x^2}} + \frac{1}{2}\frac{3}{4}\frac{5}{6} b^3 \int \frac{x^3\,dx}{\sqrt{2ax - x^2}} + \text{etc.},$$

each term of which can be integrated as in Ex. 3, Art. 31.

5. To develop $\int \frac{dx}{1+x^2}$.

Let $x < 1$, and develop $(1+x^2)^{-1}$; we thus obtain

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \text{etc.}$$

$$\therefore \int \frac{dx}{1+x^2} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.} \dots + C'.$$

But $\int \frac{dx}{1+x^2} = \text{tang}^{-1} x + C.$

$$\therefore \text{tang}^{-1} x + C = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.} \dots + C'.$$

If we suppose the integral to begin when $x = 0$, we have $C = C'$, and, therefore,

$$\text{tang}^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \text{etc.}$$

This formula is not convergent for values of the arc greater than 45°, since we suppose x to be less than *unity.*

If $x > 1$, then we may write

$$(1+x^2)^{-1} = (x^2+1)^{-1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8} \dots \text{etc.}$$

$$\therefore \int = \text{tang}^{-1} x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \text{etc.} + C.$$

$x = \infty$ renders the first member equal to $\frac{\pi}{2}$, and all the terms in the second member, except C, equal to *zero.*

Therefore $C = \frac{\pi}{2}$, and

$$\text{tang}^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} \text{ etc.} \dots$$

This formula is applicable only to arcs between 45° and 90°.

Definite Integrals.

42. It has already been shown, in the first chapter, that in order to determine the value of a definite integral, when taken between two given limits, it is sufficient to substitute in the indefinite or general integral the two limiting values of the variable, and take the difference between the results.

The formula for this operation is

$$\int_{x_1}^{x_2} F'(x)\,dx = F(x_2) - F(x_1), \text{ or}$$

$$\int_{x_1}^{x_2} F'(x)\,dx = \int_{x_0}^{x_2} F'(x)\,dx - \int_{x_0}^{x_1} F'(x)\,dx;$$

x_0 being the value of x at which the integral is supposed to originate.

This formula ceases to be applicable when the function is not continuous between the limits $F(x_1)$ and $F(x_2)$, or when any intermediate value of the function is equal to infinity.

In such cases we may divide the interval $x_2 - x_1$ into such parts that the function may be continuous throughout each part, then integrate between the new limits, and take the sum of the results. Thus, if a be a value of x between x_1 and x_2, such that $F(a) = \infty$, we may integrate between the limits x_1 and a, then between a and x_2, and take the sum of the integrals; for, by the very nature of integration, we must have

$$\int_{x_1}^{x_2} F'(x)\,dx = \int_{x_1}^{a} F'(x)\,dx + \int_{a}^{x_2} F'(x)\,dx.$$

Corollary.—It is evident that we must also have

$$\int_{x_1}^{x_2} = -\int_{x_2}^{x_1}.$$

NOTE.—The student will observe that the term *limit* is not employed in the preceding remarks in its technical sense, as the value toward which a variable may be made to converge indefinitely without ever reaching it; but in its ordinary sense, as a *definite* value of the variable itself.

EXAMPLES.

1. $\int_0^1 x^m\,dx.$

We have $$\int x^m\,dx = \frac{x^{m+1}}{m+1} + C.$$

Substituting the two values of x, 1 and 0, and subtracting the results, we have

$$\int_0^1 x^m\,dx = \frac{1^{m+1} - 0^{m+1}}{m+1} = \frac{1}{m+1}.$$

2. $\int_0^\infty \frac{dx}{e^x}.$

We have $$\int \frac{dx}{e^x} = -\frac{1}{e^x} + C.$$

$$\therefore \int_0^\infty \frac{dx}{e^x} = -\frac{1}{e^\infty} + \frac{1}{e^0} = 1.$$

3. $\int_0^\infty \frac{dx}{1+x^2}.$ Ans. $\tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}.$

4. $\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2}.$

Ans. $\frac{1}{a}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}(-\infty) = \frac{\pi}{a}.$

5. $\int_0^\infty e^{-dz}\sin bz\,dz.$

From formula [Y], Art. 39, we have

$$\int e^{-az}\sin bz\,dz = -\frac{a\sin bz + b\cos bz}{a^2+b^2}\,e^{-az}.$$

Substituting in this formula the two values of z, ∞ and 0, and subtracting the results, we have

$$\int_0^{\infty} e^{-az}\sin bz\,dz = \frac{b}{a^2+b^2}.$$

6. $\displaystyle\int_0^{\infty} e^{-az}\cos bz\,dz.$

Applying formula [X], Art. 39, we have

$$\int_0^{\infty} = \frac{a}{a^2+b^2}.$$

7. $\displaystyle\int_a^b \frac{dx}{x}.$ Ans. $\log b - \log a = \log\left(\frac{b}{a}\right).$

8. $\displaystyle\int_{-a}^{-b} \frac{dx}{x}.$ Ans. $\log(-b) - \log(-a) = \log\left(\frac{b}{a}\right).$

9. $\displaystyle\int_{-a}^{b} \frac{dx}{x}.$

In this example the limits are of different signs, and the function is not continuous from b to $-a$, being equal to $-\infty$ for $x = 0$.

Taking the integral between the limits, b, 0, 0, $-a$, we have

$$\int_{-a}^{b} \frac{dx}{x} = \int_0^b \frac{dx}{x} + \int_{-a}^{0} \frac{dx}{x}$$

$$= \log(b) - \log(0) + \log(0) - \log(-a)$$

$$= \text{an indeterminate expression.}$$

In this and similar cases we may proceed as follows: Integrate first between the limits b and em, and then between the limits $-en$ and $-a$, in which m and n are arbitrary constants, and e is an infinitesimal.

We will thus obtain

$$\int_{em}^{b} \frac{dx}{x} = \log(b) - \log(e) - \log(m);$$

$$\int_{-a}^{-en} \frac{dx}{x} = \log(e) + \log(n) - \log(a).$$

$$\therefore \quad \int_{em}^{b} + \int_{-a}^{-en} = \log\left(\frac{b}{a}\right) + \log\left(\frac{n}{m}\right).$$

The sum of these two integrals is independent of e; it is dependent on the values of n and m, and is therefore indeterminate. But if we pass to the limit by making e equal to *zero*, we shall still have

$$\int_{em}^{b} + \int_{-a}^{-en} = \int_{0}^{b} + \int_{-a}^{0} = \int_{-a}^{b} \frac{dx}{x}$$

$$= \log\left(\frac{b}{a}\right) + \log\left(\frac{n}{m}\right).$$

If, finally, we suppose n equal to m, whence

$$\log\left(\frac{n}{m}\right) = \log 1 = 0,$$

we shall have, *for this particular relation between* n *and* m,

$$\int_{-a}^{b} \frac{dx}{x} = \log\left(\frac{b}{a}\right),$$

which is called the *principal value* of this definite integral.

10. Find, by substitution in the proper formulas, the values of the following definite integrals:

$$\int_0^{\infty} x^n e^{-x}\, dx. \qquad \text{Ans. } 1.\ 2.\ 3.\ \ldots\ n.$$

$$\int_0^{\infty} x^n e^{-ax}\, dx. \qquad \text{Ans. } \frac{1.\ 2.\ 3.\ \ldots\ n}{a^{n+1}}.$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx. \quad \text{Ans. } \frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}\,\frac{\pi}{2} \text{ when } n \text{ is even;}$$

$$\frac{2.4.6\ldots(n-1)}{1.3.5\ldots n} \text{ when } n \text{ is odd.}$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx. \quad \text{Ans. the same as in the last example.}$$

11. $\int_0^{\pi} \cos(x\cos\theta)\,d\theta.$

Developing $\cos(x\cos\theta)$ into a series, we have

$$\cos(x\cos\theta) = 1 - \frac{x^2\cos^2\theta}{1.2} + \frac{x^4\cos^4\theta}{1.2.3.4}\cdots + \frac{x^{2m}\cos^{2m}\theta}{1.2.3\ldots 2m} + \text{etc.}$$

Substituting this series in the given expression, it becomes

$$\int = \int_0^{\pi} d\theta - \frac{1}{1.2}\int_0^{\pi} x^2\cos^2\theta\,d\theta + \frac{1}{1.2.3.4}\int_0^{\pi} x^4\cos^4\theta\,d\theta + \cdots + \frac{1}{1.2.3\ldots 2m}\int_0^{\pi} x^{2m}\cos^{2m}\theta\,d\theta + \text{etc.} \quad (1).$$

Now we have, by Ex. 10,

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1.3.5\ldots n-1}{2.4.6\ldots n}\,\frac{\pi}{2} \text{ when } n \text{ is even.}$$

$$\therefore \int_0^{\pi} \cos^{2m}\theta\,d\theta = \frac{1.3.5\ldots 2m-1}{2.4.6\ldots 2m}\,\pi.$$

$$\therefore \int_0^{\pi} x^2\cos^2\theta\,d\theta = \frac{1}{2}\,x^2\pi;$$

$$\int_0^\pi x^4 \cos^4\theta\, d\theta = \frac{1.\,3}{2.\,4}\, x^4\,\pi;$$

$$\int_0^\pi x^6 \cos^6\theta\, d\theta = \frac{1.\,3.\,5}{2.\,4.\,6}\, x^6\,\pi;$$

" " " " " "

Substituting these values in (1), it becomes

$$\int_0^\pi \cos(x\cos\theta)\, d\theta = \pi\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.\,4^2} - \frac{x^6}{2^2.\,4^2.\,6^2}\text{ etc.}\right\}.$$

Since $\int_0^\pi \cos^n\theta\, d\theta = \int_0^\pi \sin^n\theta\, d\theta$, it follows that the above result is also the value of $\int_0^\pi \cos(x\sin\theta)\, d\theta$.

12. Find the value of

$$\int_0^\pi \sin\theta \sin(x\sin\theta)\, d\theta.$$

43. Differentiation and Integration under the Sign $\int$.

The integral of a given differential function being the limit to the sum of all the infinitesimal elements of which it is composed, we may differentiate each of these elements with respect to a new variable, and take the limit to the sum of the new set of differentials so obtained. This will evidently be the same as differentiating, with respect to the new variable, the quantity under the integral sign; or, since the sum of any number of differentials is equal to the differential of the sum of the functions, and since the limit to the sum is equal to the sum of the limits, it follows that this operation will be the same as differentiating the $\int$ itself.

All the processes, therefore, for differentiating ordinary functions will apply, without change, to integral functions.

Problem.—To differentiate with respect to x,

$$u = \int_{z_0}^{Z} F(x. z)\, dz,$$

in which z_0 and Z are functions of x.

We have

$$u = \int_{z_0}^{Z} F(x, z)\, dz = \int F(x, Z)\, dz - \int F(x, z_0)\, dz;$$

$$\frac{du}{dx} = \frac{du}{dz_0}\frac{dz_0}{dx} + \frac{du}{dZ}\frac{dZ}{dx} + \frac{du}{dx}.$$

Now, $\dfrac{du}{dz_0} = -F(x, z_0)$; $\quad \dfrac{du}{dZ} = F(x, Z)$;

$$\frac{du}{dx} = \int_{z_0}^{Z} \frac{dF(x, z)}{dx}\, dz.$$

$$\therefore \frac{d}{dx}\int_{z_0}^{Z} F(x, z)\, dz = \{F(x, Z)\}\frac{dZ}{dx}$$
$$- \{F(x, z_0)\}\frac{dz_0}{dx} + \int_{z_0}^{Z} \frac{dF(x, z)}{dx}\, dz.$$

Corollary 1.—If z_0 and Z are not functions of x, then

$$\frac{dz_0}{dx} = 0, \text{ and } \frac{dZ}{dx} = 0,$$

and we shall have

$$\frac{d}{dx}\int_{z_0}^{Z} F(x, z)\, dz = \int_{z_0}^{Z} \frac{dF(x, z)}{dx}\, dz;$$

an expression which shows that in such cases *the order of performing the differentiation and integration is immaterial.*

Corollary 2.—We have at once,

$$\text{if} \quad u = \int F(x, z)\, dz + C;$$

$$\frac{du}{dx} = \int \frac{dF(x, z)}{dx}\, dz + \frac{dC}{dx}.$$

44. If, instead of differentiating, we wish to *integrate* with respect to x the expression $\int_{z_0}^{Z} F(x, z)\,dz$, in which z is independent of x, it will be sufficient to observe that

$$\int_{x_0}^{X} dx \int_{z_0}^{Z} F(x, z)\,dz \text{ and} \int_{z_0}^{Z} dz \int_{x_0}^{X} F(x, z)\,dx$$

have the same derivative with respect to x, and as they are both *zero* when $x = x_0$, they are equal to each other.

[This is, of course, under the supposition that neither of the expressions becomes infinite or indeterminate within the given limits.]

45. By means of differentiation and integration under the integral sign we may often find the values of definite integrals with more facility than by any other process. We shall give here a few examples by way of illustration.

1. We have already found

$$\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}\cdot$$

If we differentiate this expression n times with reference to a, we shall have

$$\int_0^{\infty} x^{n-1} e^{-ax}\,dx = \frac{1.\,2.\,3\,\ldots\,(n-1)}{a^n} \qquad (1).$$

Making $a = 1$ in this equation, we have

$$\int_0^{\infty} x^{n-1} e^{-x}\,dx = 1.\,2.\,3\,\ldots\,(n-1) \qquad (2).$$

Designating this integral by $\Gamma(n)$, we have

$$\Gamma(n) = 1.\,2.\,3\,\ldots\,(n-1), \text{ and}$$

$$\int_0^{\infty} e^{-ax} x^{n-1}\,dx = \frac{1}{a^n}\int_0^{\infty} e^{-x} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n} \qquad (3).$$

If we put $e^{-x} = y$, we shall have

$$\text{for } x = 0,\ y = 1; \quad \text{for } x = \infty,\ y = 0,$$

$$\text{and } x = \log\left(\frac{1}{y}\right), \quad dx = -\frac{dy}{y}.$$

$$\therefore \int_0^\infty e^{-x}\, x^{n-1}\, dx = -\int_1^0 \left\{\log\frac{1}{y}\right\}^{n-1} dy$$

$$= \int_0^1 \left\{\log\frac{1}{y}\right\}^{n-1} dy = \Gamma(n) \qquad (4).$$

If we put $x^n = z$, then

$$e^{-x} = e^{-z^{\frac{1}{n}}}, \quad x^{n-1}\, dx = \frac{dz}{n}, \text{ and}$$

$$\int_0^\infty e^{-x}\, x^{n-1}\, dx = \frac{1}{n}\int_0^\infty e^{-z^{\frac{1}{n}}} dz = \Gamma(n) \qquad (5).$$

If in (5) we put $n = \frac{1}{2}$, we shall have

$$\int_0^\infty e^{-x}\, x^{-\frac{1}{2}}\, dx = 2\int_0^\infty e^{-z^2}\, dz = \Gamma\left(\frac{1}{2}\right).$$

The second member of this equation having no obvious numerical significance, we shall determine the real value of this integral by another process.

$$\text{Assume } K = \int_0^\infty e^{-z^2}\, dz.$$

Then, since the value of the integral is independent of the variable, we shall also have

$$K = \int_0^\infty e^{-y^2}\, dy;$$

and by multiplication,

$$K^2 = \int_0^\infty e^{-y^2}\, dy \int_0^\infty e^{-z^2}\, dz.$$

This may be written (since y and z are independent of each other)

$$K^2 = \int_0^\infty \int_0^\infty e^{-(y^2+z^2)}\, dy\, dz.$$

Now, let $y = tz$, whence $dy = z\, dt$.

Then we shall have

$$K^2 = \int_0^\infty \int_0^\infty e^{-(t^2 z^2+z^2)}\, z\, dt\, dz = \int_0^\infty dt \int_0^\infty e^{-z^2(1+t^2)}\, z\, dz$$

$$= \int_0^\infty dt \int_0^\infty \frac{e^{-z^2(1+t^2)}\, 2z(1+t^2)\, dz}{2(1+t^2)}.$$

If in the last term we make $z^2(1+t^2) = u$, it will become

$$\int_0^\infty \frac{e^{-z^2(1+t^2)}\, 2z(1+t^2)\, dz}{2(1+t^2)} = \int_0^\infty \frac{e^{-u}\, du}{2(1+t^2)}$$

$$= \frac{1}{2(1+t^2)} \int_0^\infty e^{-u}\, du$$

$$= \frac{1}{2(1+t^2)} \quad \text{[See Ex. 2, Art. 42].}$$

$$\therefore \quad K^2 = \int_0^\infty \frac{dt}{2(1+t^2)} = \frac{1}{2} \int_0^\infty \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left(\tan^{-1} \infty - \tan^{-1} 0\right)$$

$$= \frac{\pi}{4}.$$

$$\therefore \quad K = \int_0^\infty e^{-z^2}\, dz = \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \qquad (6);$$

and therefore $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

By means of (6) we can find the value of

$$u = \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx.$$

Differentiating this expression with respect to a, we have

$$\frac{du}{da} = -2a \int_0^\infty \frac{dx}{x^2} e^{-\left(x^2 + \frac{a^2}{x^2}\right)}.$$

If we put $x = \frac{a}{z}$, we shall find, since the value of the integral is independent of the variable,

$$\frac{du}{da} = -2u.$$

$$\therefore \frac{du}{u} = -2\,da; \text{ whence } \log\frac{u}{c} = -2a, \text{ and}$$

$$u = ce^{-2a}.$$

To determine the value of c, put $a = 0$; then

$$u = c = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ by (6).}$$

$$\therefore \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} e^{-2a} \tag{7}$$

If we integrate $\int_0^\infty e^{-ax} dx = \frac{1}{a}$ between the limits $a = b$ and $a = c$, we shall have

$$\int_0^\infty \frac{e^{-bx} - e^{-cx}}{x} dx = \log\left(\frac{c}{b}\right) \tag{8}$$

45′. The integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the **Eulerian integral of the second species.** It has a remarkable relation to the integral $\int_0^1 x^{(b-1)} (1-x)^{n-1} dx$, which is called the **Eulerian** integral of the **first species**, the names being derived from that of the mathematician who first studied them.

To develop this relation. If in the expression

$$\int_0^1 x^{(b-1)}(1-x)^{n-1}\,dx,$$

we put $x = \dfrac{1}{1+z}$, we shall have

$$\int_0^1 x^{(b-1)}(1-x)^{n-1}\,dx = \int_0^\infty \frac{z^{n-1}\,dz}{(1+z)^{n+b}},$$

since $z=\infty$ when $x=0$, and $-\int_\infty^0 = +\int_0^\infty$.

Now, applying formula [A] to this last integral, we have, as may be readily seen,

$$\int_0^\infty \frac{z^{n-1}\,dz}{(1+z)^{n+b}} = \frac{1.\,2.\,3\,\ldots\,(n-1)}{(n+b-n)(\ldots)(\ldots)(n+b-2)(n+b-1)}$$

$$= \frac{1.\,2.\,3\,\ldots\,(n-1)\,\ldots\,1.\,2.\,3\,\ldots\,(b-1)}{1.\,2.\,3\,\ldots\,(b-1)b\,\ldots\,(n+b-1)}.$$

But $1.\,2.\,3\ldots(n-1) = \Gamma(n)$; $1.\,2.\,3\ldots(b-1) = \Gamma(b)$;

$$1.\,2.\,3\ldots(n+b-1) = \Gamma(n+b).$$

Hence, $$\int_0^\infty \frac{z^{n-1}\,dz}{(1+z)^{n+b}} = \frac{\Gamma(n)\,\Gamma(b)}{\Gamma(n+b)},$$

or by substitution

$$\int_0^1 x^{b-1}(1-x)^{n-1}\,dx$$

$$= \frac{\left\{\int_0^\infty x^{n-1}e^{-x}\,dx\right\}\left\{\int_0^\infty x^{b-1}e^{-x}\,dx\right\}}{\int_0^\infty x^{n+b-1}e^{-x}\,dx} \qquad (9).$$

If in this equation we make $b = 2$, and $n = 1$, we shall have

$$\int_0^1 x\,dx = \frac{\left\{\int_0^\infty e^{-x}\,dx\right\}\left\{\int_0^\infty x\,e^{-x}\,dx\right\}}{\int_0^\infty x^2\,e^{-x}\,dx}.$$

But $$\int_0^1 x\,dx = \frac{1}{2}, \text{ and} \int_0^\infty e^{-x}\,dx = 1;$$

$\therefore$ by substitution and reduction,

$$\int_0^\infty x^2\,e^{-x}\,dx = 2\int_0^\infty x\,e^{-x}\,dx.$$

GEOMETRICAL APPLICATIONS OF THE INTEGRAL CALCULUS.

CHAPTER VII.

RECTIFICATION OF CURVES. QUADRATURE OF PLANE AREAS.

46. Problem.—*To determine a formula for the length of an arc of a plane curve; i. e.*, to rectify it.

Let $y = F(x)$ be the equation to a plane curve, and designate by s the length of the arc.

Then, since [Diff. Cal., Art. 106]

$$ds = \sqrt{dx^2 + dy^2},$$

we shall have

$$s = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int dx\sqrt{1 + \frac{dy^2}{dx^2}} = \int dy\sqrt{1 + \frac{dx^2}{dy^2}}.$$

The value of this integral, taken between the limits x_1, x_2, or y_1, y_2, will evidently be the length of the arc between the points whose coördinates are $x_1 y_1$ and $x_2 y_2$.

Designating this arc by S, we have

$$S = \int_{x_1}^{x_2} dx \sqrt{1 + \frac{dy^2}{dx^2}} \qquad (1), \quad \text{or}$$

$$S = \int_{y_1}^{y_2} dy \sqrt{1 + \frac{dx^2}{dy^2}} \qquad (2).$$

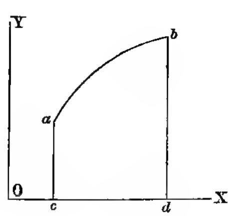
Corollary.—If the curve be referred to *polar* coördinates, we shall have [Diff. Cal., Art. 106]

$$S = \int_{r_1}^{r_2} dr \sqrt{1 + r^2 \frac{d\theta^2}{dr^2}} \qquad (3), \quad \text{or}$$

$$S = \int_{\theta_1}^{\theta_2} d\theta \sqrt{r^2 + \frac{dr^2}{d\theta^2}} \qquad (4).$$

47. Problem.—*To determine a formula for the area included between an arc, its two extreme ordinates, and the axis of abscissas.*

Fig. 29.



Designating the required area by A, and by x_1 x_2 the abscissas of its two extreme points, we have [Diff. Cal., Art. 107]

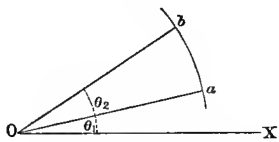
$$d A = ydx.$$

$$\therefore \quad A = \int_{x_1}^{x_2} ydx \qquad (5).$$

In applying this formula the value of y must be expressed in terms of x from the equation of the curve.

Corollary.—If the curve be referred to *polar* coördinates, then we shall have [Diff. Cal., Art. 108]

Fig. 30.



$$dA = \frac{1}{2} r^2 d\theta.$$

$$\therefore \quad A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta \quad (6),$$

will be the formula for the area included between two radii-vectores and their corresponding arc.

48. Problem.—*To determine a formula for the length of an arc of a curve of double curvature.*

We have [Diff. Cal., Art. 137]

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \qquad (a)$$

$$= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}.$$

$$\therefore \quad S = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} \quad (7), \quad \text{or}$$

$$S = \int_{y_1}^{y_2} dy \sqrt{1 + \left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2} \quad (8), \quad \text{or}$$

$$S = \int_{z_1}^{z_2} dz \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} \quad (9).$$

Corollary.—The formulas for passing from rectangular to polar coördinates are

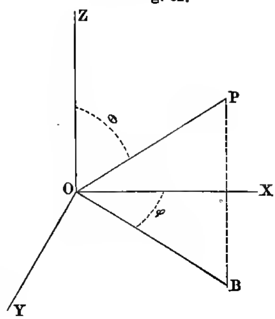
$$x = r \sin\theta \cos\phi;$$

$$y = r \sin\theta \sin\phi;$$

$$z = r \cos\theta,$$

in which θ is the angle between the radius-vector and the axis of z; and ϕ is the angle between the projection of the radius on the plane XY and the axis of x.

Fig. 31.



Differentiating these equations with respect to θ, and substituting the values of dx, dy, and dz in (a), we obtain

$$ds = d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2};$$

$$\therefore \quad S = \int_{\theta_1}^{\theta_2} d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2} \quad (10).$$

Examples.

49. 1. The length of the parabola.

The equation of the parabola is $y^2 = 2px$; whence

$$\frac{dx}{dy} = \frac{y}{p}. \qquad \therefore \quad S = \frac{1}{p}\int_{y_1}^{y_2} dy \sqrt{y^2 + p^2}.$$

Integrating according to formula (C), we have, by taking the integral between the limits y_1 and y_2,

$$S = \frac{1}{2p}\left(y_2\sqrt{p^2 + y_2^2}\right) + \frac{p}{2}\log\{y_2 + \sqrt{p^2 + y_2^2}\}$$

$$- \frac{1}{2p}\left(y_1\sqrt{p^2 + y_1^2}\right) - \frac{p}{2}\log\{y_1 + \sqrt{p^2 + y_1^2}\}$$

$$= \frac{1}{2p}\left(y_2\sqrt{p^2 + y_2^2} - y_1\sqrt{p^2 + y_1^2}\right) + \frac{p}{2}\log\left\{\frac{y_2 + \sqrt{p^2 + y_2^2}}{y_1 + \sqrt{p^2 + y_1^2}}\right\}.$$

If $y_1 = 0$, then we shall have for that portion of the curve included between the vertex and the point whose ordinate is y,

$$S' = \frac{y}{2p}\sqrt{p^2 + y^2} + \frac{p}{2}\log\left\{\frac{y + \sqrt{p^2 + y^2}}{p}\right\}.$$

2. The length of the ellipse.

The equation of the ellipse is

$$a^2 y^2 + b^2 x^2 = a^2 b^2; \quad \text{whence}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{b^2 x}{a^2 y}\right)^2 = \left\{\frac{a^2(1-e^2)x}{a^2 y}\right\}^2$$

$$= \frac{(1-e^2)^2 x^2}{(1-e^2)(a^2-x^2)} = \frac{(1-e^2)x^2}{a^2-x^2}; \quad \text{and}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(1-e^2)x^2}{a^2-x^2} = \frac{a^2 - e^2 x^2}{a^2 - x^2}.$$

$$\therefore \quad S = \int_{x_1}^{x_2}\left(\frac{(a^2 - e^2 x^2)}{a^2 - x^2}\right)^{\frac{1}{2}} dx.$$

To obtain this integral, we have, by expansion,

$$(a^2 - e^2 x^2)^{\frac{1}{2}} = a - \frac{1}{2}\,\frac{e^2 x^2}{a} - \frac{1}{2}\,\frac{1}{4}\,\frac{e^4 x^4}{a^3} - \frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,\frac{e^6 x^6}{a^5}, \text{ etc.}$$

$$\text{Hence} \int\left(\frac{a^2 - e^2 x^2}{a^2 - x^2}\right)^{\frac{1}{2}} dx = a\int\frac{dx}{\sqrt{a^2 - x^2}} - \frac{1}{2}\,\frac{e^2}{a}\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}$$

$$- \frac{1}{2}\,\frac{1}{4}\,\frac{e^4}{a^3}\int\frac{x^4\,dx}{\sqrt{a^2-x^2}}$$

$$- \frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,\frac{e^6}{a^5}\int\frac{x^6\,dx}{\sqrt{a^2-x^2}}, \text{ etc.}$$

Putting $x = az$, we shall have

$$S = a\left\{\int \frac{dz}{\sqrt{1-z^2}} - \frac{1}{2}e^2\int\frac{z^2\,dz}{\sqrt{1-z^2}} - \frac{1}{2}\,\frac{1}{4}\,e^4\int\frac{z^4\,dz}{\sqrt{1-z^2}}\right.$$
$$\left. - \frac{1}{2}\,\frac{1}{4}\,\frac{3}{6}\,e^6\int\frac{z^6\,dz}{\sqrt{1-z^2}},\ \text{etc.}\right\},$$

each term of which can be obtained by the formula for $\int\frac{x^m\,dx}{\sqrt{1-x^2}}$ [Art. 31, Ex. 1].

To find the length of the elliptic quadrant, we must integrate between the limits $x = 0$ and $x = a$, or $z = 0$ and $z = 1$. But

$$\int_0^1 \frac{z^m\,dz}{\sqrt{1-z^2}} = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{m-1}{m}\,\frac{\pi}{2}.$$

$$\therefore \int_0^1 \frac{z^6\,dz}{\sqrt{1-z^2}} = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\frac{\pi}{2};$$

$$\int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}} = \frac{1}{2}\,\frac{3}{4}\,\frac{\pi}{2};$$

$$\int_0^1 \frac{z^2\,dz}{\sqrt{1-z^2}} = \frac{1}{2}\,\frac{\pi}{2};\quad \int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2}.$$

We have, therefore,

$$A\ Quadrant = \frac{a\pi}{2}\left\{1 - \frac{1}{2^2}e^2 - \frac{1}{2^2}\,\frac{3}{4^2}\,e^4\right.$$
$$\left. - \frac{1}{2^2}\,\frac{3^2}{4^2}\,\frac{5}{6^2}\,e^6 - \text{etc.}\right\},$$

and the entire circumference of the ellipse =

$$2\pi a\left\{1 - \frac{1}{2^2}e^2 - \frac{1}{2^2}\,\frac{3}{4^2}\,e^4 - \frac{1}{2^2}\,\frac{3^2}{4^2}\,\frac{5}{6^2}\,e^6 - \text{etc.}\right\}.$$

3. The length of the cycloid.

We have previously found for the cycloid [Diff. Cal., Art. 94]

$$\frac{dy}{dx} = \sqrt{\frac{2r - x}{x}}.$$

$$\therefore \quad 1 + \frac{dy^2}{dx^2} = \frac{2r}{x}, \quad \text{and}$$

$$S = \int_{x_1}^{x_2} \sqrt{\frac{2r}{x}}\, dx = 2\sqrt{2r}\{\sqrt{x_2} - \sqrt{x_1}\}.$$

To determine the length of one-half the cycloid, we must integrate between the limits $x_1 = 0$ and $x_2 = 2r$.

We thus obtain semi-cycloidal arc $= 4r$, and, therefore, entire cycloid $= 8r = 4(2r) =$ four times the diameter of the generating circle.

4. The length of the spiral of Archimedes. We have

$$r = a\theta; \;\; dr = a d\theta; \;\; \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = \frac{1}{a}\sqrt{a^2 + r^2}.$$

$$\therefore \quad S = \frac{1}{a}\int_{r_1}^{r_2} \sqrt{a^2 + r^2}\, dr.$$

This integral may be obtained from that of the length of the parabola by changing y into r and p into a.

5. The length of the logarithmic spiral. We have

$$r = a^\theta; \;\; dr = a^\theta \log a\, d\theta; \;\; \frac{dr}{d\theta} = a^\theta \log a.$$

$$\therefore \quad S = \int_{\theta_1}^{\theta_2} d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \int_{\theta_1}^{\theta_2} d\theta \sqrt{r^2(1 + \log^2 a)}$$

$$= \int_{\theta_1}^{\theta_2} a^\theta\, d\theta \sqrt{1 + \log^2 a} = \frac{1}{\log a}\sqrt{1 + \log^2 a}\,\{a^{\theta_2} - a^{\theta_1}\}$$

$$= \sqrt{1 + m^2}\,\{r_2 - r_1\},$$

in which m is the modulus of the system of logarithms.

6. The length of the catenary.

The equation of the catenary being

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

we shall find $ds = \dfrac{y\,dy}{\sqrt{y^2 - c^2}}$.

$$\therefore\quad S = \int_{y_1}^{y_2} \frac{y\,dy}{\sqrt{y^2 - c^2}} = \sqrt{y_2^2 - c^2} - \sqrt{y_1^2 - c^2}.$$

If the arc begin at the point where $y = c$, we shall have

$$S = \sqrt{y_2^2 - c^2}.$$

7. The length of the helix.

The equations of the helix are

$$y = a \sin \frac{z}{ma}; \qquad x = a \cos \frac{z}{ma}.$$

$$\therefore\quad \frac{dy}{dz} = \frac{1}{m} \cos \frac{z}{ma}; \qquad \frac{dx}{dz} = -\frac{1}{m} \sin \frac{z}{ma}.$$

Substituting these values in formula (9), Art. 48, we have

$$S = \int_{z_1}^{z_2} dz \sqrt{1 + \frac{1}{m^2} \sin^2 \frac{z}{ma} + \frac{1}{m^2} \cos^2 \frac{z}{ma}}$$

$$= \int_{z_1}^{z_2} dz \sqrt{1 + \frac{1}{m^2}}$$

$$= \frac{1}{m} z \sqrt{1 + m^2}$$

if the arc begin at the point where $z = 0$.

8. The length of the loxodrome, or rhumb-line, described by a vessel whose direction makes a constant angle with the meridian.

The equations of this curve are

$$\sqrt{x^2+y^2}\,\{e^{a\tan^{-1}\frac{y}{x}}+e^{-a\tan^{-1}\frac{y}{x}}\}=2r,$$

$$x^2+y^2+z^2=r^2.$$

Changing to polar coördinates, as in Art. 48, Cor., we have

$$S=\int_{\theta_1}^{\theta_2} d\theta\,\frac{\sqrt{1+a^2}}{a}$$

$$=\frac{\sqrt{1+a^2}}{a}\,(\theta_2-\theta_1),$$

the values of θ being the co-latitudes of the two terminal points.

9. The area of the parabola.

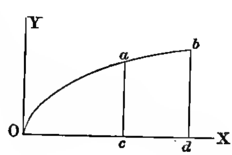
We have $y^2=2px$, whence $y=\sqrt{2px}$.

$$\therefore\ A=\int_{x_1}^{x_2}\sqrt{2px}\,dx$$

$$=\frac{2}{3}\sqrt{2p}\,\{x_2^{\frac{3}{2}}-x_1^{\frac{3}{2}}\}=\frac{2}{3}\,\{y_2x_2-y_1x_1\}$$

$= \dfrac{2}{3}\left\{\begin{array}{l}\text{difference of rectangles}\\ \text{described on the abscissas}\\ \text{and ordinates of the two}\\ \text{extreme points.}\end{array}\right\}$

Fig. 32.



If $x_1=0,\ y_1=0$, then

$$A=\frac{2}{3}\,xy;\ \text{ or}$$

the area of the portion $Oabd$ is equal to two-thirds of the rectangle whose sides are Od and bd.

10. The area of the ellipse.

We have $y = \frac{b}{a} \sqrt{a^2 - x^2}$.

$$\therefore \quad A = \frac{b}{a} \int_{x_1}^{x_2} (a^2 - x^2)^{\frac{1}{2}} \, dx.$$

Integrating by parts, we obtain

$$\int (a^2 - x^2)^{\frac{1}{2}} \, dx = x \sqrt{a^2 - x^2} + \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}}$$

$$= x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} - \int (a^2 - x^2)^{\frac{1}{2}} dx.$$

$$\therefore \int (a^2 - x^2)^{\frac{1}{2}} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + C;$$

and $A = \frac{b}{a} \left\{ \frac{1}{2} x_2 \sqrt{a^2 - x_2^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x_2}{a} \right.$

$$\left. - \frac{1}{2} x_1 \sqrt{a^2 - x_1^2} - \frac{1}{2} a^2 \sin^{-1} \frac{x_1}{a} \right\}.$$

Integrating between the limits $x = 0$ and $x = a$, we have for the *area of a quadrant,*

$$A = \frac{ab}{2} \sin^{-1} 1 = \frac{\pi ab}{4};$$

and for the area of the entire ellipse,

$A = \pi ab = \sqrt{\pi a^2 . \pi b^2} =$ mean of circles described on the two axes.

11. The area of a cycloid.

We have $A = \int y \, dx = xy - \int x \, dy$.

Now, $dy = \sqrt{\frac{2r - x}{x}} \, dx$.

$$\text{Hence, } \int x\,dy = \int x\sqrt{\frac{2r-x}{x}}\,dx = \int \sqrt{2rx-x^2}\,dx$$

$$= \int \sqrt{r^2-(x-r)^2}\,dx$$

$$= \frac{x-r}{2}\sqrt{2rx-x^2} + \frac{r^2}{2}\sin^{-1}\frac{x-r}{r} + C,$$

[by the last example].

$$\therefore\ A = x_2y_2 - x_1y_1 - \frac{x_2-r}{2}\sqrt{2rx_2-x_2^2} - \frac{r^2}{2}\sin^{-1}\frac{x_2-r}{r}$$

$$+ \frac{x_1-r}{2}\sqrt{2rx_1-x_1^2} + \frac{r^2}{2}\sin^{-1}\frac{x_1-r}{r}.$$

Taking $x_1 = 0$, $y_1 = 0$, $x_2 = 2r$, $y_2 = \pi r$, we find for one-half the area of the cycloid

$A = \dfrac{3\pi r^2}{2}$, and for the whole area, $A = 3\pi r^2 =$ three times the area of the generating circle.

12. The area of the catenary.

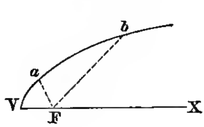
$$\text{We have } A = \int y\,dx = \int \frac{1}{2}c\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}dx.$$

$$= \frac{1}{2}c^2\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = c\sqrt{y^2-c^2}.$$

This is the expression for the area included between the axis of x, the curve, and the two ordinates, $y_1 = c$, $y_2 = y$.

13. The area included between the arc and two focal radii of the parabola.

Fig. 33.



The polar equation of the parabola is

$$r = \frac{p}{1+\cos\theta} = \frac{p}{2\cos^2\frac{1}{2}\theta}.$$

$$\therefore\ A = \int \frac{1}{2} r^2 d\theta$$

$$= \frac{1}{4} p^2 \int \frac{d\frac{\theta}{2}}{\cos^4 \frac{1}{2}\theta} = \frac{1}{4} p^2 \int \frac{d\frac{\theta}{2}(1 + \operatorname{tang}^2 \theta)}{\cos^2 \frac{1}{2}\theta}$$

$$= \frac{1}{4} p^2 \int \sec^2 \frac{\theta}{2} d\frac{\theta}{2} + \frac{1}{4} p^2 \int \operatorname{tang}^2 \frac{\theta}{2} \sec^2 \frac{\theta}{2} d\frac{\theta}{2}$$

$$= \frac{1}{4} p^2 \left\{ \operatorname{tang} \frac{\theta_2}{2} + \frac{1}{3} \operatorname{tang}^3 \frac{\theta_2}{2} - \operatorname{tang} \frac{\theta_1}{2} - \frac{1}{3} \operatorname{tang}^3 \frac{\theta_1}{2} \right\}$$

between the limits θ_1 and θ_2.

If $\theta_1 = 0$, then we have for the area VFb,

$$A = \frac{1}{4} p^2 \left\{ \operatorname{tang} \frac{\theta}{2} + \frac{1}{3} \operatorname{tang}^3 \frac{\theta}{2} \right\}.$$

14. The area of the spiral of Archimedes.

We have $r = a\theta$.

$$\therefore\ A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2} a^2 \int_{\theta_1}^{\theta_2} \theta^2 d\theta = \frac{1}{6} a^2 \{ \theta_2^3 - \theta_1^3 \}.$$

If $\theta_1 = 0$, then

$$A = \frac{1}{6} a^2 \theta^3 = \frac{1}{6} a^2 \frac{r^3}{a^3} = \frac{1}{6} \frac{r^3}{a}.$$

15. The length and area of the lemniscata, $r^2 = a^2 \cos 2\theta$.

16. The length and area of the ellipse, the pole being at the focus.

17. The length and area of the hyperbolic spiral, $r\theta = a$.

18. The length of the semi-cubical parabola, $ay^2 = x^3$.

19. The area of the cissoid, $y^2(2a - x) = x^3$.

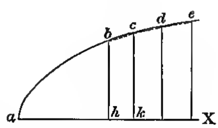
20. The area of the curve, $r = a \sin 3\theta$.

CHAPTER VIII.

QUADRATURE OF CURVED SURFACES. CUBATURE OF SOLIDS.

50. Problem.— *To determine a general formula for the area of a surface of revolution.*

Fig. 34.



Let $abcde$ be the generating curve, and let the axis of X be taken as the axis of revolution.

If the curve be revolved about this axis, it will describe a surface of revolution, and each side of the polygon formed by the chords ab, bc, etc., will describe the surface of a conical frustum. It is plain that *the surface of revolution will be the limit to the sum of the surfaces described by the chords, as these are indefinitely increased in number.*

Designating by x and y the coördinates of a point b,

by $x + \Delta x$, $y + \Delta y$ those of c,

and by Δch the chord bc,

we shall have

Surface described by $bc = 2\pi \left\{ \dfrac{2y + \Delta y}{2} \right\} \Delta ch$. Hence,

$$\left\{ \begin{array}{c} \text{The whole surface} \\ \text{described by the polygon} \end{array} \right\} = \Sigma\, 2\pi \left\{ \frac{2y + \Delta y}{2} \right\} \Delta ch; \text{ and}$$

$\therefore$ The surface of revolution = limit of surface described by polygon

$$= \lim \Sigma\, 2\pi \left\{ \frac{2y + \Delta y}{2} \right\} \Delta ch$$

$$= 2\pi \int y ds,$$

in which s represents the arc of the generating curve.

Designating the area by A, and substituting for ds its value $\sqrt{dx^2 + dy^2}$, we have

$$A = 2\pi \int y \sqrt{dx^2 + dy^2} \qquad (1),$$

the integration to be made between the given limits.

51. Problem.—*To determine a general formula for the volume of a solid of revolution.*

It is evident that the volume in question is the limit to the sum of the conical frustums of which it is composed. Now we have (see last figure)

Frustum described by $bckh$

$$= \frac{1}{3} \pi \Delta x \{y^2 + (y + \Delta y)^2 + y(y + \Delta y)\}$$

$$= \frac{1}{3} \pi \Delta x \{3y^2 + 3y\,\Delta y + (\Delta y)^2\}.$$

$$\therefore \text{ Entire Solid } = \lim \Sigma \frac{1}{3} \pi \Delta x \{3y^2 + 3y\,\Delta y + (\Delta y)^2\}$$

$$= \int \pi y^2\, dx;$$

or designating the solid by V, we have

$$V = \pi \int y^2\, dx \qquad (2).$$

Corollary.—In any solid of revolution the section perpendicular to the axis is a circle. If, in any solid which has an axis, the section perpendicular to this axis be a curve whose equation may be written $y = f(x)$, then it is easily seen that if the *area* of this section be designated by $F(x)$, the volume will be given by the formula

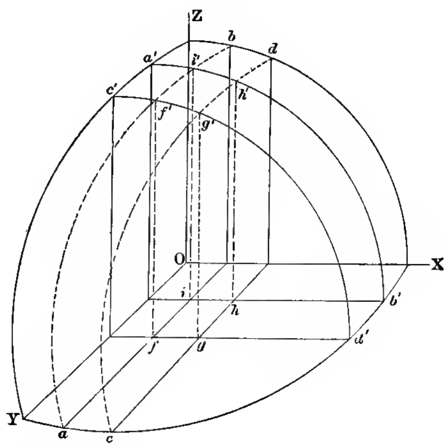
$$V = \int F(x)\, dx \qquad (3),$$

the integration to be made between given limits.

52. Problem.—*To find the area of any curved surface given by its equation.*

Let $F(x, y, z) = 0$ be the equation of the surface referred to three rectangular coördinate axes, and let $f(x, y) = 0$ be the equation of its intersection with the plane of XY. If we suppose the surface to be intersected by two series of planes parallel to XZ and YZ, respectively, it will be divided up

Fig. 35.



into a series of *curvilinear quadrilaterals.* Connecting the angular points of these quadrilaterals by straight lines, we shall have inscribed within the curved surface a polyhedral surface, each face of which is a rectilinear quadrilateral; and the given surface will evidently be the limit to the sum of all these rectilinear quadrilaterals which are inscribed within it. If, also, the distances between the intersecting planes are Δx and Δy, respectively, then the area of the projection of each of the quadrilaterals upon the plane XY will be the product of Δx and Δy.

Again, the area of the projection of a surface upon a given plane being equal to the area of the surface itself

multiplied by the cosine of the angle between its plane and the plane of projection, it follows that

$$\Delta x\, \Delta y = \Delta a \cos \alpha, \quad \text{or} \quad \Delta a = \frac{\Delta x\, \Delta y}{\cos \alpha},$$

in which Δa designates the area of the quadrilateral, and α is the angle between its plane and that of XY.

Now, passing to the limits, we shall have

$$\lim \Delta a = da; \quad \lim \Delta x\, \Delta y = dx\, dy;$$

$$\lim \cos \alpha = \cos t = \text{cosine}$$

of angle between the plane of XY and the tangent plane to the surface at the point x, y, z

$$= \frac{1}{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}} \quad \text{[Diff. Cal., Art. 141].}$$

Hence, $da = dx\, dy \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}$.

The area of the surface will be found by integrating this expression *twice*, once between the limits x_1, x_2, and once between the limits $y_1 = f(x_1)$, and $y_2 = f(x_2)$, derived from the equation $f(x, y) = 0$.

The integration with respect to x, in effecting which we may regard y and dy as constant, will give us the *limit to the sum of the quadrilaterals contained between two planes parallel to XZ*, and the second integration will give the entire surface.

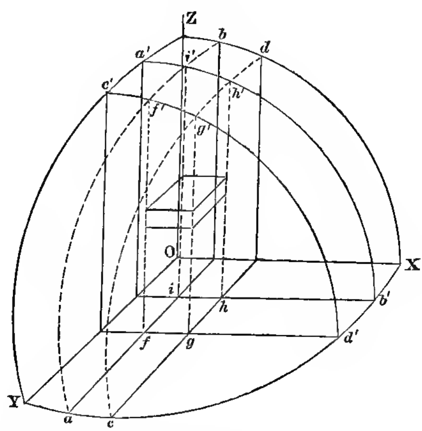
We therefore have

$$A = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dx\, dy \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} \qquad (4).$$

53. Problem.—*To determine the volume of any solid bounded by curved surfaces.*

If we suppose the body to be intersected by three systems of planes, respectively parallel to the coördinate planes, it will thus be divided into a series of rectangular parallelopipeds and of oblique polyhedral figures, one face of each of the latter being a curvilinear element of the surface of

Fig. 36.



the given body. If the number of intersecting planes be indefinitely increased, the limit to the sum of these oblique figures will evidently be *zero*, and *the limit to the sum of the parallelopipeds will be the volume of the body itself.*

Designating the distances between the intersecting planes by Δx, Δy, Δz, and by Δv the volume of a parallelopiped, we shall have

$$\Delta v = \Delta x \, \Delta y \, \Delta z,$$

and, passing to the limit,

$$dv = dx \, dy \, dz, \text{ or}$$

$$V = \int dv = \int dx \, dy \, dz.$$

If this expression be integrated with respect to x, supposing y and z to remain constant, we shall obtain the volume of a section of the body bounded by four planes, two of them parallel to XZ, and two parallel to YX. Then, integrating with respect to y, we shall have the sum of all the sections contained between the two last mentioned planes; and, finally, integrating with respect to z, we shall have the entire volume.

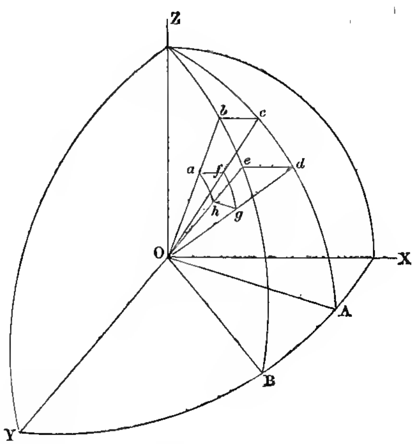
We may therefore write

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx\, dy\, dz \qquad (5).$$

in which the integration is to be performed in the order indicated above.

54. Problem. — *To determine the volume of a body when referred to polar coördinates.*

Fig. 37.



Let the body be intersected by a series of planes passing through OZ, then by conical surfaces described by the revolution of Oc, Od, etc., and finally by a system of spheres having their common center at O. We shall thus have the

body divided up into a series of elementary parts, such as $abcdefgh$, and the volume of the body will be the limit to the sum of these parts.

Designating the coördinates of g by $r = Og$, $\theta = gOA$, $\phi = AOX$, we shall have

$$gd = \Delta r;\quad gf = r\Delta\theta;\quad gh = r\cos\theta\Delta\phi;\ \text{ and}$$

$$\text{volume } abc\text{——}h = r^2\Delta r\cos\theta\Delta\theta\Delta\phi;$$

or, passing to the limits, and taking the sum,

$$V = \int r^2\,dr\cos\theta\,d\theta\,d\phi.$$

It is easily seen, as in the last proposition, that this expression must be integrated three times, and we shall have, finally,

$$V = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2} r^2\,dr\cos\theta\,d\theta\,d\phi \qquad (6).$$

NOTE.—This formula is based on the assumption that abc——h is a parallelopiped. This is not rigidly correct, but the difference between abc——h and the parallelopiped is evidently an infinitesimal which disappears in taking the limit to the sum.

NOTE.—In *double* and *triple* integration between limits it is necessary to pay special attention to the order of integration, if the limits of the different variables be dependent upon each other. It is obvious that a change in the order of integration may necessitate a change in the limits between which the integral is to be taken.

55. EXAMPLES.

1. The surface generated by the revolution of a semi-ellipse about its major axis. We have

$$y = \frac{b}{a}\sqrt{a^2 - x^2};\ \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}.$$

$$\therefore\ A = 2\pi \frac{b}{a}\int_{x_1}^{x_2} \sqrt{a^2 - e^2 x^2}\, dx$$

$$= 2\pi \frac{ba}{e}\int_{x_1}^{x_2} \sqrt{1 - \frac{e^2 x^2}{a^2}}\, d\,\frac{ex}{a}$$

$$= 2\pi \frac{ba}{e}\left\{ \frac{ex_2}{2a}\sqrt{1 - \frac{e^2 x_2^2}{a^2}} + \frac{1}{2}\sin^{-1}\frac{ex_2}{a}\right.$$

$$\left. - \frac{ex_1}{2a}\sqrt{1 - \frac{e^2 x_1^2}{a^2}} - \frac{1}{2}\sin^{-1}\frac{ex_1}{a}\right\}.$$

Taking the integral between the limits $-a$ and $+a$, we have, for the entire surface of the ellipsoid,

$$A = 2\pi \frac{ba}{e}\left\{ e\sqrt{1 - e^2} + \sin^{-1} e \right\}$$

$$= 2\pi a^2 \left\{ (1 - e^2) + \frac{\sqrt{1 - e^2}}{e} \sin^{-1} e \right\}.$$

If $e = 0$, the ellipsoid becomes a sphere, $\dfrac{\sin^{-1} e}{e} = 1$, and

$$A = 4\pi a^2.$$

2. The surface generated by the revolution of a cycloid about its base.

Here we have $\dfrac{dy}{dx} = \sqrt{\dfrac{2r - x}{x}}$; $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{\dfrac{2r}{x}}$,

and since in the formula for the surface of revolution y is the distance from a point on the curve to the axis, we may write $y = 2r - x$.

Hence, $A = 2\pi \displaystyle\int_{x_1}^{x_2} (2r - x)\sqrt{\frac{2r}{x}}\, dx$

$$= 2\pi\sqrt{2r}\left\{ \begin{array}{c} 4r\,(x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) \\ -\frac{2}{3}\,(x_2^{\frac{3}{2}} + x_1^{\frac{3}{2}}) \end{array} \right\}.$$

Taking twice the integral between the limits 0 and $2r$, we have, for the entire surface,

$$A = \frac{64}{3}\pi r^2.$$

3. The surface generated by the revolution of an ellipse about its minor axis.

Changing the formula to $2\pi\int x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$, since the axis of y is now the axis of revolution, we have

$$A = 2\pi a^2\left\{1+\frac{1-e^2}{2e}\log\left\{\frac{1+e}{1-e}\right\}\right\},$$

for the entire surface of the ellipsoid.

4. The surface generated by the revolution of a cycloid about its axis.

We have, as in example 2,

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{2r}{x}}.$$

$$\therefore\ A = 2\pi\int y\sqrt{\frac{2r}{x}}\,dx,$$

$$= 4\pi y\sqrt{2rx} - 4\pi\sqrt{2r}\int\sqrt{2r-x}\,dx.$$

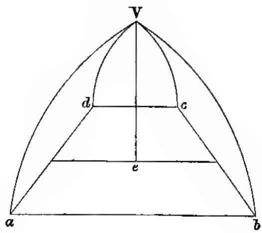
Taking the integral between the limits $x=0$ and $x=2r$, we have, for the entire surface,

$$A = 8\pi r^2\left(\pi - \frac{4}{3}\right).$$

5. Find the area of the surface formed by the intersection of two equal circular cylinders at right angles to each other.

Let the origin be at the point of intersection of the axes, and let the axes of the two cylinders be taken as the axes of Y and Z. Then the equations of the cylinders will be

Fig. 38.



$$x^2 + z^2 = a^2, \text{ and}$$

$$x^2 + y^2 = a^2.$$

Hence, $\dfrac{dz}{dx} = -\dfrac{x}{z}$ and $\dfrac{dz}{dy} = 0$.

$$\therefore \quad A = \int\int dy\, dx \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} = a\int\int \frac{dy\, dx}{\sqrt{a^2 - x^2}}.$$

This must be integrated, first between the limits $y = 0$ and $y = \sqrt{a^2 - x^2}$, and then between the limits $x = 0$ and $x = a$. We thus obtain

$$A = a\int_0^a dx = a^2,$$

which is the expression for one-eighth the entire surface.

6. The volume generated by the revolution of an ellipse about its major axis.

We have at once,

$$V = \pi\int_{x_1}^{x_2} y^2\, dx = \pi\int_{x_1}^{x_2} \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{4}{3}\pi ab^2 = \frac{2}{3}\pi b^2 . 2a$$

(between the limits $x = -a$ and $+a$)

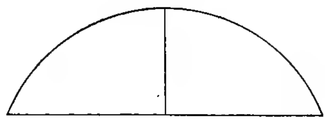
$$= \frac{2}{3} \text{ circumscribing cylinder.}$$

7. The volume of the paraboloid of revolution.

Ans. $\frac{1}{2}$ circumscribing cylinder.

8. The volume of the solid formed by revolving a cycloid about its base.

Fig. 39.



We have

$$\frac{dy}{dx} = \sqrt{\frac{2r - x}{x}},$$

the origin being at the vertex.

It will facilitate the integration to transfer the origin to the middle point of the base, and reverse the axes. For this purpose we have

$$y = 2r - x, \quad x = 2r - y.$$

Hence, $\dfrac{dy}{dx} = \dfrac{d(2r - x)}{d(2r - y)} = \sqrt{\dfrac{y}{2r - y}}$, and

$$dx = dy\sqrt{\frac{y}{2r - y}}.$$

$$\therefore \quad V = \pi\int_{y_1}^{y_2} y^2\sqrt{\frac{y}{2r - y}}\,dy = \pi\int_{y_1}^{y_2} y^{\frac{5}{2}}(2r - y)^{-\frac{1}{2}}\,dy.$$

Integrating by formula (A), we obtain

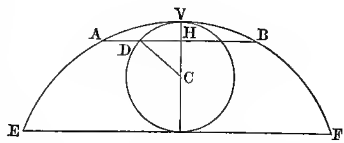
$$V = \pi(2r - y)^{\frac{1}{2}}\left\{\frac{1}{3}y^{\frac{5}{2}} + \frac{5}{6}ry^{\frac{3}{2}} + \frac{5}{2}r^2y^{\frac{1}{2}}\right\}$$
$$- \frac{5}{2}\pi r^3 \operatorname{versin}^{-1}\frac{y}{r} + C.$$

Taking twice this integral between the limits $y = 0$ and $y = 2r$, we obtain for the entire volume,

$$V = 5\pi^2r^3 = 2\pi r \times \frac{5}{2}\pi r^2$$
$$= \frac{5}{8} \text{ circumscribing cylinder.}$$

9. The volume of the solid described by the revolution of a cycloid about it axis.

Fig. 40.



This integral can not be conveniently obtained by the use of rectangular co-ordinates; we shall therefore transform to polar co-ordinates, the pole being at C, the center of the generating circle, when the latter is in the position indicated in the figure.

Let A be a point on the curve, $VH = x$, $AH = y$, $DC = r$, and $DCH = \theta$. Then we shall have

$$VH = x = VC - CH = r - r\cos\theta;$$

$$AH = y = AD + DH = VD + DH = r\theta + r\sin\theta.$$

$$\therefore\ V = \pi\int y^2\,dx = \pi r^3\int(\theta + \sin\theta)^2 d\,(1 - \cos\theta).$$

Integrating this expression between the limits $\theta = 0$ and $\theta = \pi$, we have for the entire volume,

$$V = \pi r^3\left\{3\,\frac{\pi^2}{2} - \frac{8}{3}\right\}.$$

10. The volume of the ellipsoid with three unequal axes.

Every plane section perpendicular to the axis of x being an ellipse, we may determine the volume by the application of the formula $\int F(x)\,dx$.

The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the semi-axes of an elliptic section at the distance x from the origin are $\frac{c}{a}\sqrt{a^2-x^2}$, $\frac{b}{a}\sqrt{a^2-x^2}$. The area of this section is $F(x)=\frac{\pi bc}{a^2}(a^2-x^2)$. Hence,

$$V=\int_{x_1}^{x_2} F(x)\,dx=\frac{\pi bc}{a^2}\int_{x_1}^{x_2}(a^2-x^2)\,dx=\frac{4}{3}\pi bca$$

(between the limits $x=-a$ and $x=+a$)

$$=\frac{2}{3}\text{ circumscribing cylinder.}$$

11. The volume of the paraboloid with elliptic section.

The equation to this paraboloid is

$$bz^2+ay^2=2abx.$$

Finding the area of the elliptic section, as in the preceding case, and integrating between the limits $x=0$ and $x=x$, we have

$$V=\frac{1}{2}\text{ circumscribing cylinder.}$$

12. The surface of the tri-rectangular spherical triangle.

The equation of the surface is

$$x^2+y^2+z^2=r^2.$$

Hence, $\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}=\frac{r}{z}$, and

$$A=\int\int dy\,dx\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}=\int\int dy\,dx\,\frac{r}{z}$$

$$=\int\int\frac{r\,dy\,dx}{\sqrt{r^2-x^2-y^2}}.$$

This expression must be integrated once between the limits $y=0$ and $y=\sqrt{r^2-x^2}$, and once between the limits $x=0$ and $x=r$. We thus obtain, since

$$\int\frac{dy}{\sqrt{r^2-x^2-y^2}}=\sin^{-1}\frac{y}{\sqrt{r^2-x^2}}=\frac{\pi}{2}\text{ when } y=\sqrt{r^2-x^2},$$

$$A=\frac{1}{2}\pi r\int_0^r dx=\frac{1}{2}\pi r^2.$$

13. The volume of the tri-rectangular spherical sector.

We have $V=\int\int\int dx\,dy\,dz,$

in which the limits of z are 0 and $\sqrt{r^2-x^2-y^2}$;

those of y are 0 and $\sqrt{r^2-x^2}$;

those of x are 0 and r.

$$\therefore\quad V=\int\int dx\,dy\sqrt{r^2-x^2-y^2}.$$

Now $$\int\sqrt{r^2-x^2-y^2}\,dy=\frac{1}{2}y\sqrt{r^2-x^2-y^2}$$

$$+\frac{1}{2}(r^2-x^2)\sin^{-1}\frac{y}{\sqrt{r^2-x^2}}$$

$$=\frac{1}{4}\pi(r^2-x^2)\text{ when } y=\sqrt{r^2-x^2}.$$

$$\therefore\quad V=\frac{1}{4}\pi\int(r^2-x^2)\,dx=\frac{1}{6}\pi r^3\text{ when } x=r.$$

14. The volume of the hemisphere, referred to polar coördinates.

We have $V=\int\int\int r^2\,dr\cos\theta\,d\theta\,d\phi,$

and the limits of r are 0 and r;

those of θ are 0 and $\frac{1}{2}\pi$;

those of ϕ are 0 and 2π.

Integrating between these limits, we have

$$V = \frac{1}{3} r^3 \int\int \cos\theta \, d\theta \, d\phi = \frac{1}{3} r^3 \sin\frac{1}{2}\pi \int d\phi$$

$$= \frac{1}{3} r^3 \, 2\pi = \frac{2}{3}\pi r^3.$$

55′. When the formula $V = \int\int\int dx\, dy\, dz$ leads to the integration of complicated algebraic expressions, we may frequently simplify the operation by integrating with respect to one of the variables, and then transferring the other two to polar coördinates. For this purpose let the given expression be integrated with respect to z. Then we shall have

$$V = \int\int z \, dx \, dy.$$

The formulas for transformation being

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we shall have

$$dx = \frac{x}{r}\, dr - y d\theta \qquad (1),$$

$$dy = \frac{y}{r}\, dr + x d\theta \qquad (2).$$

Now, in effecting the double integration, when we wish to determine x, we suppose y to be constant, and therefore $dy = 0$; and if we wish to determine y, we suppose x to be constant, and therefore $dx = 0$.

Adopting the latter hypothesis, we have

$$0 = \frac{x}{r}\, dr - y d\theta,$$

$$dy = \frac{y}{r}\, dr + x d\theta;$$

whence, by eliminating $d\theta$,

$$y dy = r dr \qquad (a).$$

From this equation we infer that when $dy = 0$, dr is also equal to *zero*. We have then, from (1),

$$dx = -y d\theta \qquad (b).$$

Multiplying (*a*) and (*b*) together, we have

$$dx\, dy = -r dr\, d\theta,$$

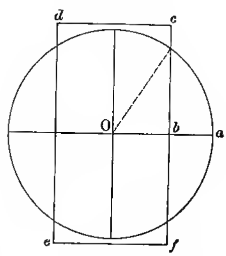
and, therefore, neglecting the $-$ sign, which merely indicates the direction in which θ is reckoned from the axis of X, the formula for integration becomes

$$V = \int\int z\, d\theta\, r\, dr.$$

EXAMPLE 1.—A sphere is intersected by a right circular cylinder whose axis passes through the center of the sphere: find the intercepted volume.

Taking the origin at the center of the sphere, and the axis of the cylinder as the axis of z, we have for the equations of the surfaces,

Fig. 41.



$$x^2 + y^2 + z^2 = a^2;$$

$$x^2 + y^2 = b^2;$$

a being the radius of the sphere, and b being the radius of the cylinder.

Now it is easily seen from the figure that the limits of z are 0 and $\sqrt{a^2 - r^2}$; those of θ are 0 and 2π, and those of r are 0 and b.

We therefore have
$$\begin{aligned} V &= \int\int z\, d\theta\, r\, dr \\ &= \int\int \sqrt{a^2 - r^2}\, d\theta\, r\, dr \\ &= 2\pi \int \sqrt{a^2 - r^2}\, r\, dr \\ &= \frac{2\pi}{3} \{a^3 - (a^2 - b^2)^{\frac{3}{2}}\}. \end{aligned}$$

This is the expression for that portion of the volume which lies above the plane of XY.

The entire volume is twice this, or

$$\frac{4\pi}{3}\{a^3 - (a^2 - b^2)^{\frac{3}{2}}\}.$$

EXAMPLE 2.—A sphere is intersected by a cylinder, the radius of whose base is half that of the sphere, and whose axis bisects the radius of the sphere at right angles: find the intercepted volume.

The equations of the surfaces are

$$x^2 + y^2 + z^2 = a^2; \quad x^2 + y^2 = ax.$$

The limits of r are 0 and $a \cos \theta$; those of θ are 0 and $\frac{1}{2}\pi$.

$$\text{Ans. } \frac{1}{4} \text{ entire volume} = \frac{a^3}{3}\left\{\frac{\pi}{2} - \frac{2}{3}\right\}.$$

INTEGRATION OF FUNCTIONS OF TWO OR MORE VARIABLES.

CHAPTER IX.

DIFFERENTIAL EXPRESSIONS.

56. Hitherto our attention has been directed exclusively to the integration of functions of a single variable, comprehended under the general form $F(x)\,dx$.

We propose now to examine those differential expressions which involve two or more variables, and to investigate the general methods of integrating them, *i. e.*, of determining the functions of which they are the total differentials.

Let us consider the expression

$$Mdx + Ndy \qquad (1),$$

in which $\qquad M = \phi(x, y), \quad N = \psi(x, y).$

We observe, in the first place, that (1) is not necessarily the *exact* differential of any function of x and y. For, if it were so, designating this function by u, we should have

$$M = \frac{du}{dx}, \quad N = \frac{du}{dy};$$

and since $\qquad \dfrac{d^2u}{dx\,dy} = \dfrac{d^2u}{dy\,dx}$ [Diff. Cal., Art. 72],

we must also have $\qquad \dfrac{dM}{dy} = \dfrac{dN}{dx} \qquad (2)$

If, then, M and N do not satisfy the indentity represented by (2), there is no function of x and y of which (1) is the exact differential.

We shall see, however, that, if this condition is fulfilled by M and N, the function in question does exist, and also how it may be determined.

For this purpose, we remark, that since M is the partial derivative of the function with respect to x, the function itself must be included in the general, or indefinite, integral of Mdx with respect to x, y being regarded as a constant. It is, then, included in the expression

$$\int_{x_0}^{x} Mdx + v,$$

in which v is an unknown function of y.

It remains to determine the function in such a manner that its derivative with respect to y shall be N.

Now this derivative is

$$\int_{x_0}^{x} \frac{dM}{dy}\,dx + \frac{dv}{dy}, \text{ or its equivalent}$$

$$\int_{x_0}^{x} \frac{dN}{dx} dx + \frac{dv}{dy}; \text{ or, in fine,}$$

$$\psi(x, y) - \psi(x_0, y) + \frac{dv}{dy}.$$

Therefore, $\psi(x, y) - \psi(x_0, y) + \dfrac{dv}{dy} = N = \psi(x, y)$, and

$$\therefore \quad -\psi(x_0, y) + \frac{dv}{dy} = 0;$$

or, by transposing an integrating with respect to y,

$$v = \int_{y_0}^{y} \psi(x_0, y)\, dy + C.$$

We have thus demonstrated that when equation (2) is satisfied there exists a function of x and y of which (1) is the exact differential, and the foregoing analysis shows that the integration may be exhibited in the following equation:

$$\int (Mdx + Ndy) = \int_{x_0}^{x} \phi(x, y)\, dx + \int_{y_0}^{y} \psi(x_0, y)\, dy + C \quad (3).$$

57. Let us take, in the next place, a function of three variables,

$$Mdx + Ndy + Pdz \quad (4);$$

in which $M = \phi(x, y, z)$; $N = \psi(x, y, z)$; $P = \chi(x, y, z)$.

If (4) is the exact differential of any function of x, y, and z, then, as in the preceding case, we must have

$$\frac{dM}{dy} = \frac{dN}{dx}; \quad \frac{dM}{dz} = \frac{dP}{dx}; \quad \frac{dN}{dz} = \frac{dP}{dy} \quad (5).$$

These conditions being fulfilled, the function in question is necessarily included in the indefinite integral of Mdx with respect to x, and it will therefore be given by the formula

$$\int_{x_0}^{x} Mdx + v$$

in which v is an unknown function of y and z.

The partial derivatives of the function with reference to y and z are, respectively, N and P. Therefore,

$$N = \int_{x_0}^{x} \frac{dM}{dy}dx + \frac{dv}{dy} = \int_{x_0}^{x} \frac{dN}{dx}dx + \frac{dv}{dy}$$
$$= N - \psi(x_0, y, z) + \frac{dv}{dy};$$

$$P = \int_{x_0}^{x} \frac{dM}{dz}dx + \frac{dv}{dz} = \int_{x_0}^{x} \frac{dP}{dx}dx + \frac{dv}{dz}$$
$$= P - \chi(x_0, y_0, z) + \frac{dv}{dz}.$$

$$\therefore \quad \frac{dv}{dy} = \psi(x_0, y, z); \qquad \frac{dv}{dz} = \chi(x_0, y_0, z);$$

and therefore, by the last case

$$v = \int_{y_0}^{y} \psi(x_0, y, z)\, dy + \int_{z_0}^{z} \chi(x_0, y_0, z)\, dz.$$

The function in question, then, exists necessarily when the conditions (5) are satisfied, and is given by the formula

$$\int (Mdx + Ndy + Pdz) = \int_{x_0}^{x} \phi(x, y, z)\, dx$$
$$+ \int_{y_0}^{y} \psi(x_0, y, z)\, dy + \int_{z_0}^{z} \chi(x_0, y_0, z)\, dz + C \quad (6).$$

58. When a given differential expression satisfies equations (2) or (5), which are called the **conditions of integrability,** it is said to be an **exact differential**; and its integral may be found by the application of formulas (3) or (6).

It is obvious that in these formulas we may take $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$, and this observation leads to the following simple

Rule for integrating exact differentials.—*Integrate* Mdx *with respect to* x; *then integrate all the terms in* dy *which do not contain* x; *and lastly, integrate all the terms in* dz *which do not contain* x *or* y. *The sum of the results will be the required integral.*

59. It is to be understood, of course, that the variables are entirely independent of each other, and when this is the case, no *expression* which does not satisfy the condition (2) or (5) can be completely integrated in finite terms. If there is a known relation between the variables, it may render possible an integration which otherwise could not be effected.

60. EXAMPLES.

1. Integrate $\dfrac{dx}{\sqrt{1+x^2}} + a\,dx + 2by\,dy$. We have

$$M = a + \frac{1}{\sqrt{1+x^2}}; \quad N = 2by; \quad \frac{dM}{dy} = 0 = \frac{dN}{dx}.$$

The expression is therefore an exact differential, and its integral is

$$\int = \int\left(a + \frac{1}{\sqrt{1+x^2}}\right)dx + \int 2by\,dy$$
$$= ax + \log\{x + \sqrt{1+x^2}\} + by^2 + C.$$

2. Integrate $(a^2y + x^3)\,dx + (b^3 + a^2x)\,dy$.

We have $M = a^2y + x^3$; $N = b^3 + a^2x$;

$$\frac{dM}{dy} = a^2 = \frac{dN}{dx}.$$

$$\therefore \int = \int(a^2y + x^3)\,dx + \int b^3\,dy = a^2xy + \frac{1}{4}x^4 + b^3y + C.$$

3. Integrate $\dfrac{dx}{\sqrt{x^2+y^2}} + \dfrac{dy}{y} - \dfrac{x\,dy}{y\sqrt{x^2+y^2}}$.

We have $M = \dfrac{1}{\sqrt{x^2+y^2}}$; $N = \dfrac{1}{y}\left\{1 - \dfrac{x}{\sqrt{x^2+y^2}}\right\}$;

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

$$\therefore \int = \int \frac{dx}{\sqrt{x^2+y^2}} + \int \frac{dy}{y}$$

$$= \log \left\{ \frac{x + \sqrt{x^2+y^2}}{y} \right\} + \log y + C$$

$$= \log \{ c\,(x + \sqrt{x^2+y^2}) \}.$$

4. Integrate $(3xy^2 - x^2)\,dx + (3x^2y - 6y^2 - 1)\,dy$.

Ans. $\dfrac{3x^2y^2}{2} - \dfrac{x^3}{3} - 2y^3 - y + C.$

5. Integrate $(6x^2y + x^3)\,dx + (2a^2x - 3by^3)\,dy$.

Ans. No Integral.

6. Integrate $\dfrac{2\,dx}{\sqrt{x^2-y^2}} - \dfrac{2x\,dy}{y\sqrt{x^2-y^2}}$.

We shall find $\dfrac{dM}{dy} = \dfrac{dN}{dx}$.

$$\therefore \int = \int \frac{2\,dx}{\sqrt{x^2-y^2}} = 2 \log \left\{ \frac{c\,(x + \sqrt{x^2-y^2})}{y} \right\}.$$

7. Integrate $\dfrac{y\,dx}{a-z} + \dfrac{x\,dy}{a-z} + \dfrac{xy\,dz}{(a-z)^2}$.

We have $\dfrac{dM}{dy} = \dfrac{1}{a-z} = \dfrac{dN}{dx}$; $\dfrac{dM}{dz} = \dfrac{y}{(a-z)^2} = \dfrac{dP}{dx}$;

$$\frac{dN}{dz} = \frac{x}{(a-z)^2} = \frac{dP}{dy}.$$

Hence the expression is integrable, and we have

$$\int = \frac{xy}{a-z} + C.$$

8. Integrate $\frac{a\,dx - b\,dy}{z} + \frac{by - ax}{z^2}\,dz.$

This expression satisfies the conditions of integrability, and we have

$$\int = \frac{ax - by}{z}.$$

9. Integrate $\frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\,dx - x\,dz}{x^2 + z^2} + z\,dz.$

Ans. $\sqrt{x^2 + y^2 + z^2} + \tan^{-1}\frac{x}{z} + \frac{1}{2}z^2 + C.$

CHAPTER X.

DIFFERENTIAL EQUATIONS.

61. The integration of a differential equation which involves two or more variables, consists in the determination of a finite equation of which the given equation is a consequence.

Up to the present time this operation has been effected in but a comparatively small number of cases, and we shall, after establishing certain general principles relating to the subject, confine ourselves to the examination of some of the simplest forms in which the integration has been accomplished.

62. Integration by Maclaurin's Formula.—Let us take the general equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots \frac{d^my}{dx^m}\right) = 0 \qquad (1),$$

which is said to be of the m^{th} order, and of the degree indicated by the highest exponent with which any derivative is affected.

From this equation we shall obtain, by solution, the value of $\frac{d^m y}{dx^m}$ in terms of x, y, $\frac{dy}{dx} \cdots \frac{d^{m-1}y}{dx^{m-1}}$; and, by successive differentiation, the values of the derivatives superior to the m^{th} will be given in terms of the same quantities.

The integral of (1) will be a finite equation between y and x, which may be written $y = f(x)$; and by expanding this function of x by Maclaurin's formula, we shall have

$$y = (y) + \left(\frac{dy}{dx}\right)\frac{x}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{x^2}{1.\,2}$$

$$+ \cdots \left(\frac{d^{m-1}y}{dx^{m-1}}\right)\frac{x^{m-1}}{1.\,2.\,3 \ldots (m-1)} + \text{etc.} \quad (2).$$

If, in this general development, we replace all the derivatives of superior order to $m - 1$ by their values taken from equation (1) and its derivatives after making $x = 0$, the resulting value of y will evidently satisfy (1), and the *general* value of y will be embraced in all the particular values which can be derived from this development. If, then, all possible values of y could be developed by this formula, equation (2) would give us the complete solution of the problem of integration; but as Maclaurin's formula will not give us the development of every function of x, *we can obtain* in this manner *only those values of* y *for which none of the derivatives become* infinite *or* indeterminate *for the particular value* zero *of* x.

63. Integration by Taylor's Formula.—If we suppose $y_0 = f(x_0)$ and $y = f(x_0 + h)$, we shall have $h = x - x_0$, and therefore Taylor's formula may be written

$$y = y_0 + (x - x_0)\frac{dy}{dx} + \frac{(x - x_0)^2}{1.\,2}\frac{d^2y}{dx^2}$$

$$+ \cdots + \frac{(x - x_0)^{m-1}}{1.\,2.\,3 \ldots (m-1)}\frac{d^{m-1}y}{dx^{m-1}}, \text{ etc.} \quad (3).$$

Substituting in this equation the values of the derivatives superior to the m^{th}, found by placing $x = x_0$ in (1) and its derivatives, we shall obtain a value of y which will evidently satisfy (1), and from which (1) might, reciprocally, be obtained by differentiation.

As is the case with the development by Maclaurin's formula, so, in this development, equation (3) would give us the complete solution of the problem of integration provided none of the derivatives became infinite or indeterminate for the value x_0 of x; and this *arbitrary* value of x might always be so selected that all the derivatives would remain finite if they depended only on x_0, but as they also depend on the corresponding value y_0 of y, it may occur that a certain function $y = f(x)$, while satisfying (1), will render certain derivatives infinite or indeterminate whatever may be the value of x_0.

It follows from the above that neither (2) nor (3) can give *all* the functions which satisfy (1). Within the limits in which they are sufficiently convergent they will serve to give approximate values of the function of which (1) is the differential equation.

Equation (3) is called the **general integral** of (1), and (2) is only a *particular* integral corresponding to the case in which $x_0 = 0$.

64. Equation (3) satisfies the given differential equation, whatever may be the values of the m quantities

$$y_0, \ \frac{dy}{dx} \cdots \frac{d^{m-1}y}{dx^{m-1}};$$

and since these all disappear in performing the m differentiations which will lead from (3) to (1), it follows that they are *arbitrary constants.*

Therefore, *the general integral of a differential equation of the* m^th^ *order contains* m *arbitrary constants, which are no other than the values of* y *and its first* m — 1 *derivatives for the particular value* x_0 *of* x.

Conversely, *every finite equation between* x *and* y, *which includes* m *arbitrary constants and satisfies* (1), *is the general integral of* (1).

For, if we develop the value of y deduced from such an equation, the first m coëfficients of the development will include x_0 and the m constants; these coëfficients may have any value, whatever be the value of x_0; they may then be regarded as entirely arbitrary; and as the succeeding coëfficients depend on them *according* to the law indicated by equation (1), the entire development can not differ from that given by equation (3). Whence the truth of the proposition is apparent.

65. When in the general integral of a differential equation we give particular values to one or more of the arbitrary constants, the result is called a **particular integral.**

When we can satisfy a differential equation by a finite equation which is not included in the general integral, this finite equation is called a **singular solution,** or singular integral. We shall devote a subsequent chapter to the consideration of singular solutions.

66. We have said that every differential equation is a consequence of its *finite,* or **primitive,** equation; but it is easy to see that it is not always a *necessary,* or *direct,* consequence of this equation. For, since the primitive contains m arbitrary constants, if we differentiate it n times, we shall have $n+1$ equations from which n of the constants may be eliminated, and the resulting equation may be very different from the given one; and by varying the number and method of eliminations, we may obtain as many new differential equations as we please, each of which will necessarily satisfy the primitive, while none of them can be deduced directly from it.

If, however, *we eliminate by any number of different processes the same constants, the results of all the eliminations will be identical with each other.*

For, let us suppose that by varying the method we arrive at the two following equations of the order m:

$$\frac{d^m y}{dx^m} = F\left(x, y, \frac{dy}{dx} \cdots \frac{d^{m-1} y}{dx^{m-1}}\right),$$

$$\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx} \cdots \frac{d^{m-1} y}{dx^{m-1}}\right).$$

These functions, F and f, being the m^{th} derivatives of y, are necessarily equal to each other, whatever may be the value of x. Let $y_0, \left(\frac{dy}{dx}\right)_0, \cdots \left(\frac{d^{m-1} y}{dx^{m-1}}\right)_0$, be the values of y, $\frac{dy}{dx} \cdots \frac{d^{m-1} y}{dx^{m-1}}$ for the value x_0 of x. Then, whatever be the value of x_0, the two expressions

$$F\left\{x_0, y_0, \cdots \left(\frac{d^{m-1} y}{dx}\right)_0\right\}, \quad f\left\{x_0, y_0, \cdots \left(\frac{d^{m-1} y}{dx}\right)_0\right\},$$

are necessarily equal to each other for all values of y_0, $\left(\frac{dy}{dx}\right)_0$, etc., and the uneliminated constants; consequently, all of these expressions must enter in the same manner into the two functions F and f, thereby rendering them identical. Whence it follows that *in whatever way we eliminate the same constants we shall in all cases arrive at identical results.*

67. The last proposition leads to the following **important consequences**, viz:

That, *every differential equation of the order* m *can be deduced from* m *equations of the order* m — 1, *each of which contains* one *arbitrary constant; from* $\frac{m(m-1)}{1.\,2}$ *equations of the order* m — 2, *each of which contains* two *arbitrary constants;* and so on.

For let $y = F(x)$ be a finite equation, and let us eliminate one constant between itself and its first derivative; then eliminate one constant between the new equation so

obtained and its first derivative; and so on until m constants have disappeared. The final equation thus obtained will be the differential equation of the m^{th} order, and, from what precedes, this and each of the other equations will be identical with those which we would have obtained by any other process of eliminating the same constants. But the *order* of elimination determines the *forms* of the resulting differential equations, and the number of these equations of any particular order, as n, will evidently be equal to the number of combinations of m quantities taken n in a set. Whence the proposition as above stated.

68. It follows from the foregoing analysis that we may integrate an equation of the m^{th} order by determining, if possible, its m *first integrals*—the first integral of a differential equation of the order m being a differential of the order $m-1$. We shall thus obtain m equations containing x, y $\ldots \frac{d^{m-1}y}{dx^{m-1}}$, each of which contains one arbitrary constant, and by eliminating the derivatives we shall have a finite equation involving x, y, and m constants, which will therefore be the general integral of the given differential equation.

CHAPTER XI.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE.

69. The most general form of these equations is

$$Qdy + Pdx = 0, \text{ or } Q\frac{dy}{dx} + P = 0,$$

in which P and Q are functions of x and y.

It may always be solved by the general method of development by Taylor's or Maclaurin's formula. In some cases the resulting development can be summed in finite terms, but usually the summation is not possible.

EXAMPLE.

$$\frac{dy}{dx} + ay + bx^3 = 0.$$

By differentiation we obtain

$$\frac{d^2 y}{dx^2} + a\frac{dy}{dx} + 3bx^2 = 0,$$

$$\frac{d^3 y}{dx^3} + a\frac{d^2 y}{dx^2} + 6bx = 0,$$

$$\frac{d^4 y}{dx^4} + a\frac{d^3 y}{dx^3} + 6b = 0,$$

" " " " "

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$$\frac{d^{4+m} y}{dx^{4+m}} + a\frac{d^{3+m} y}{dx^{3+m}} = 0.$$

Making $x = 0$, we have

$$\left(\frac{dy}{dx}\right) = -a y_0; \quad \left(\frac{d^2 y}{dx^2}\right) = -a\left(\frac{dy}{dx}\right) = a^2 y_0;$$

$$\left(\frac{d^3 y}{dx^3}\right) = -a^3 y_0; \quad \left(\frac{d^4 y}{dx^4}\right) = a^4 y_0 - 6b;$$

$$\left(\frac{d^5 y}{dx^5}\right) = -a^5 y_0 + 6ab \cdots \left(\frac{d^{4+m} y}{dx^{4+m}}\right) = -a\left(\frac{d^{3+m} x}{dx^{3+m}}\right).$$

Hence, by substitution in Maclaurin's formula,

$$y = y_0\left\{1 - ax + \frac{a^2 x^2}{1.\,2} - \frac{a^3 x^3}{1.\,2.\,3}, \text{ etc.}\right\}$$

$$-6b\left\{\frac{x^4}{1.\,2.\,3.\,4} - \frac{ax^5}{1.\,2.\,3.\,4.\,5} + \frac{a^2 x^6}{1.\,2.\,3.\,4.\,5.\,6} - \text{etc.}\right\};$$

or, by substitution [Diff. Cal., Art. 53] of the value of e^{-ax},

$$y = y_0 e^{-ax} - \frac{6b}{a^4}\left\{e^{-ax} - 1 + ax - \frac{a^2x^2}{1.2} + \frac{a^3x^3}{1.2.3}\right\};$$

and, observing that $y_0 - \dfrac{6b}{a^4}$ is constant,

$$y = ce^{-ax} + \frac{6b}{a^4}\left\{1 - ax + \frac{a^2x^2}{1.2} - \frac{a^3x^3}{1.2.3}\right\}.$$

70. If Q be a function of y only, and P a function of x only, or if by any transformation the given equation can be converted into an equivalent one, each of whose terms is a function of but one of the variables, the integral may be found by integrating, by known processes, each of the terms separately, and taking the sum of the results. This method, which is much resorted to, is called *the separation of the variables*, and we shall devote the next chapter to its consideration.

71. Of factors by which the first member of an equation may be rendered an exact differential.—If the first member of the equation $Qdy + Pdx = 0$, were an exact differential, it would be necessary and sufficient that the integral should be a constant in order to satisfy the equation.

The condition of integrability of $Qdy + Pdx$ is, as we have seen, $\dfrac{dQ}{dx} = \dfrac{dP}{dy}$. If this condition be not fulfilled, the expression is not an exact differential; nevertheless, there always exists, as we shall now demonstrate, a *factor* by the introduction of which into the given expression, the latter will be transformed into an exact differential.

For this purpose, let us observe that the integral of the given equation contains an arbitrary constant c; that the given equation results from the elimination of this constant between the integral and its first derivative; and that, however the elimination be effected, the result obtained will give the value of $\dfrac{dy}{dx}$ identical with that derived from the solution of the given equation $Qdy + Pdx = 0$.

This granted, let the primitive be represented by

$$\phi(x, y) = c; \text{ whence } \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy = 0.$$

The value of $\frac{dy}{dx}$ derived from this equation being identical with that derived from $Qdy + Pdx = 0$, we must have

$$\frac{\frac{d\phi}{dx}}{\frac{d\phi}{dy}} = \frac{P}{Q}.$$

Substituting the value of P derived from this equation in $Qdy + Pdx = 0$, we obtain

$$\frac{Q}{\frac{d\phi}{dy}}\left\{\frac{d\phi}{dy}dy + \frac{d\phi}{dx}dx\right\} = 0;$$

and therefore, *if we multiply the given equation by* $\frac{1}{Q}\frac{d\phi}{dy}$ it will reduce to

$$\frac{d\phi}{dy}dy + \frac{d\phi}{dx}dx = 0,$$

each term of which is integrable.

72. Having demonstrated the existence of a factor which will render the given equation integrable, we shall in the next place show how this factor may be determined.

Designating it by v, we must have, necessarily,

$$\frac{dvQ}{dx} = \frac{dvP}{dy}, \text{ or } Q\frac{dv}{dx} - P\frac{dv}{dy} = v\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\},$$

from which equation v may be obtained by integration.

If v should be a function of both the variables, this equation would be more difficult to integrate than the original one, and we should gain nothing by attempting to discover the factor.

If v be a function of but one of the variables, say x, its value may be easily found. In this case, since $\frac{dv}{dy} = 0$, we must have

$$Q\frac{dv}{dx} = v\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}, \text{ or}$$

$$\frac{1}{v}\,\frac{dv}{dx} = \frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\},$$

and it is obvious that *if the second member of this equation be independent of* y, v may be found by direct integration.

Designating $\frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}$ by $\phi(x)$, we have

$$\frac{1}{v}\,dv = \phi(x)\,dx; \text{ whence}$$

$$\log v = \int_{x_0}^{x} \phi(x)\,dx, \text{ and } v = e^{\int_{x_0}^{x}\phi(x)dx}.$$

Substituting this value of v in the given equation

$$Qdy + Pdx = 0,$$

and integrating according to the methods in Chapter, IX, we have, for the complete integral of the given equation,

$$\int_{x_0}^{x} Pe^{\int_{x_0}^{x}\phi(x)dx}dx + \int_{y_0}^{y} Qdy = C.$$

We do not multiply the term Qdy by v, because we make use of only those terms of Qdy which do not contain x.

73. After one factor has been discovered, it is easy to obtain others. For since v renders the expression $v(Qdy + Pdx)$ an exact differential of some function, as u, of x and y, it is evident that we shall obtain by multiplying by $\phi(u)$,

$$v\phi(u)(Qdy + Pdx) = \phi(u)\,du;$$

and, the second member of this equation being integrable, the first member is also integrable.

74. If the form to be integrated be

$$dy + Pdx = 0,$$

which is called the *linear equation of the first order*, the factor of integration is readily obtained.

In this case, since $Q = 1$ and $dQ = 0$, we shall have $\frac{dP}{dy}$ independent of y, and therefore

$$P = Xy + X_1,$$

in which X and X_1 are functions of x. The equation will then be of the form

$$dy + (Xy + X_1)\,dx = 0.$$

The factor v becomes $v = e^{\int Xdx}$, and we shall have

$$e^{\int Xdx}dy + Xye^{\int Xdx}dx + X_1 e^{\int Xdx}dx = 0;$$

whence we obtain by integration with reference to y as the first variable,

$$ye^{\int Xdx} + \int X_1 e^{\int Xdx}dx = C, \text{ or}$$

$$y = e^{-\int Xdx}\left\{C - \int X_1 e^{\int Xdx}dx\right\}.$$

75. EXAMPLES.

1. Integrate $ydx - x\,dy = 0$.

We have $\frac{dP}{dy} = \frac{dy}{dy} = 1;\quad \frac{dQ}{dx} = -\frac{dx}{dx} = -1.$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = -\frac{1}{x}\{1-(-1)\} = -\frac{2}{x}; \quad \frac{dv}{v} = -\frac{2\,dx}{x};$$

$$\log v = -2\log x = \log\frac{1}{x^2}; \quad v = \frac{1}{x^2}.$$

Multiplying the given equation by this value of v and integrating, we have

$$\int \frac{y\,dx - x\,dy}{x^2} = -\frac{y}{x} = -c. \quad \therefore\ y = cx.$$

2. Integrate $dx + (a\,dx + 2\,by\,dy)\sqrt{1+x^2} = 0.$

Here $P = 1 + a\sqrt{1+x^2}$; $Q = 2by\sqrt{1+x^2}$.

$$\therefore \quad \frac{dP}{dy} = 0; \quad \frac{dQ}{dx} = \frac{2bxy}{\sqrt{1+x^2}};$$

$$\frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\} = -\frac{x}{1+x^2};$$

$$\frac{dv}{v} = -\frac{x\,dx}{1+x^2}; \quad v = (1+x^2)^{-\frac{1}{2}}.$$

$$\therefore \quad \int = \int \frac{1 + a\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx + \int 2\,by\,dy$$

$$= ax + \log(x + \sqrt{1+x^2}) + by^2 = c;$$

and $x + \sqrt{1+x^2} = e^{c-(ax+by^2)} = Ce^{-(ax+by^2)}$.

3. Integrate $x\,dy - y\,dx = x\,dx + y\,dy$.

Ans. $\tan^{-1}\frac{y}{x} = \log\sqrt{x^2+y^2} + C$.

4. Integrate $dy + x^3 y\,dx + x^4\,dx = 0.$

This is a linear equation, and we have

$$y = e^{-\int x^3\,dx}\left\{C - \int x^4 e^{\int x^3\,dx}\,dx\right\}$$

$$= e^{-\frac{1}{4}x^4}\left\{C - \int x^4 e^{\frac{1}{4}x^4}\,dx\right\}.$$

The last term in this result may be found by development.

CHAPTER XII.

INTEGRATION BY SEPARATION OF THE VARIABLES.

76. This method consists, as has already been stated, in the transformation of the given equation into an equivalent equation, each of whose terms is a function of a single variable.

When this transformation is possible, the integral of the given equation can be found by integrating each term of the new equation, and taking the sum of the results.

We shall give several cases to which this method is applicable.

77. Case I.—Let the form be

$$X\,dy + Y\,dx = 0.$$

Dividing by XY, we have

$$\frac{dy}{Y} + \frac{dx}{X} = 0,$$

in which the variables are separated.

EXAMPLE. $$x^2\,dy + y^2\,dx = 0.$$

We have $$\frac{dy}{y^2} + \frac{dx}{x^2} = 0;$$

$$\therefore\ \frac{1}{y} + \frac{1}{x} = c, \text{ and}$$

$$x + y = cxy.$$

78. Case II.—Let the form be

$$XY\,dy + X_1 Y_1\,dx = 0.$$

Dividing by XY_1, we have

$$\frac{Y}{Y_1}\,dy + \frac{X_1}{X}\,dx = 0,$$

in which the variables are separated.

EXAMPLE. $3xy^4\,dy + (x^2y^2 + y^2)\,dx = 0$, or

$$y^2(x^2+1)\,dx + 3xy^4\,dy = 0.$$

Dividing by y^2x, we have

$$\frac{x^2+1}{x}\,dx + 3y^2\,dy = 0.$$

$\therefore$ by integration,

$$\frac{1}{2}x^2 + \log x + y^3 = c, \text{ or}$$

$$x^2 = Ce^{-(x^2+2y^3)}.$$

79. Case III.—Let the form be

$$M dx + N dy = 0,$$

in which M and N are *homogeneous* functions of x and y, of the m^{th} order; *i. e.*, functions such that if we multiply each of the variables by any quantity k, the result will be identical with that obtained by multiplying the functions themselves by k^m.

Let $y = ux$; whence $dy = u\,dx + x\,du$.

The functions M and N will be equal to x^m multiplied by functions of u, and the equation, when divided by x^m, may be written

$$F(u)\,dx + f(u)\{u\,dx + x\,du\} = 0, \text{ or}$$

$$\{F(u) + uf(u)\}\,dx + xf(u)\,du = 0, \text{ or}$$

$$\frac{dx}{x} = -\frac{f(u)\,du}{F(u) + uf(u)},$$

in which the variables are separated.

Designating the integral of the second member by $\phi(u)$, we have

$$\log x = c\phi u; \text{ whence } x = e^{c\phi(u)} = Ce^{\phi(u)},$$

$$= Ce^{\phi\left(\frac{y}{x}\right)},$$

in which ϕ is a known function.

Examples.

1. $$(ax + by)\,dx = (mx + ny)\,dy.$$

We have $y = ux$; $dy = u\,dx + x\,du$; whence, by substitution,

$$u + x\frac{du}{dx} = \frac{a + bu}{m + nu}, \text{ or } x\frac{du}{dx} = \frac{a + (b - m)u - nu^2}{m + nu}.$$

$$\therefore \quad \frac{dx}{x} = \frac{(m + nu)\, du}{a + (b - m)\, u - nu^2}, \text{ and}$$

$$\log x = \int \frac{(m + nu)\, du}{a + (b - m)\, u - nu^2}.$$

The integration of the second member may be readily effected by the methods for the integration of rational fractions.

2. $$(ax + by + m)\, dx = (px + qy + n)\, dy.$$

Assuming $x = x' + \alpha,\ y = y' + \beta;\ a\alpha + b\beta + m = 0;$

$p\alpha + q\beta + n = 0;$ we have $\alpha = \dfrac{nb - mq}{aq - bp};\ \beta = \dfrac{mp - na}{aq - bp};$

$$dx = dx';\quad dy = dy';$$

and therefore, by substitution and reduction,

$$(ax' + by')\, dx' = (px' + qy')\, dy'$$

which is homogeneous and is integrable as in the last example.

If $aq - bp = 0$, the foregoing transformation can not be effected. We then have, however, $q = \dfrac{bp}{a}$ and the given equation becomes

$$(ax + by)\,(a\, dx - p\, dy) = a\,(n\, dy - m\, dx).$$

Taking $ax + by = z$, whence $dy = \dfrac{dz - a\, dx}{b}$, we will have, by eliminating y,

$$a\, dx = \frac{(an + pz)\, dz}{an + mb + (b + p)\, z},$$

in which the variables are separated.

80. Case IV.—Let the form be the linear equation

$$dy + Xydx + X_1 dx = 0.$$

Putting $y = uv$, in which u and v are arbitrary functions of x, we have

$$u\,dv + v\,du + Xuv\,dx + X_1\,dx = 0.$$

Since u and v are arbitrary, we may assume

$$du + Xu\,dx = 0;\ \text{whence } \frac{du}{u} = -Xdx, \text{ and } u = e^{-\int Xdx}:$$

$$u\,dv + X_1\,dx = 0;\ \text{whence } e^{-\int Xdx} dv + X_1\,dx = 0, \text{ and}$$

$$v = -\int X_1\,e^{\int Xdx}\,dx + C.$$

$\therefore$ by substitution,

$$y = uv = e^{-\int Xdx}\left\{C - \int X_1 e^{\int Xdx}\,dx\right\}.$$

81. Case V.—Let the form be

$$dy + Xy\,dx = X_1\,y^{n+1}\,dx,$$

which is known as **Bernoulli's** equation.

Assuming $u = \frac{1}{y^n}$, whence $y = u^{-\frac{1}{n}}$, and

$$dy = -\frac{1}{n}\,u^{-\frac{1}{n}-1}du, \text{ we have}$$

$$du - nXu\,dx + nX_1\,dx = 0,$$

a linear equation, whose integral is, by Case IV,

$$u = \frac{1}{y^n} = e^{n\int Xdx}\left\{C - n\int X_1 e^{-n\int Xdx}\,dx\right\}.$$

82. Examples, and Geometrical Applications.

1. Integrate $xdx + ydy = my\,dx$.

Assuming $y = xz$, we have

$$\frac{dx}{x} + \frac{zdz}{1 - mz + z^2} = 0 \quad (1), \text{ or}$$

$$d \log x + \frac{1}{2}\left(\frac{2z - m}{1 - mz + z^2}\right) dz + \frac{1}{2}\,\frac{mdz}{1 - mz + z^2} = 0.$$

$\therefore$ by integration and reduction,

$$\log x\sqrt{1 - \frac{my}{x} + \frac{y^2}{x^2}} + \frac{m}{\sqrt{\frac{m^2}{4} - 1}} \log \left\{ \frac{\frac{y}{x} - \frac{m}{2} - \sqrt{\frac{m^2}{4} - 1}}{\frac{y}{x} - \frac{m}{2} + \sqrt{\frac{m^2}{4} - 1}} \right\} =$$

$$\log \sqrt{x^2 - mxy + y^2} + \frac{2m}{\sqrt{m^2 - 4}} \log \left\{ \frac{2y - mx - x\sqrt{m^2 - 4}}{2y - mx + x\sqrt{m^2 - 4}} \right\} = C.$$

This solution is real when $m > 2$. It is indeterminate when $m = 2$, but in that case (1) is readily integrable. If $m < 2$, the factors of $1 - mz + z^2$ are imaginary, and the integration must be effected by the method established for such cases.

2. Integrate $x^2ydx + y^3dx = 3xy^2dy$.

Assuming $y = vx$, we find

$$\frac{dx}{x} = \frac{3vdv}{1 - 2v^2}.$$

$$\therefore \log\frac{x}{c} = 3\int\frac{vdv}{1-2v^2} = \frac{3}{4}\log\left\{\frac{1}{1-2v^2}\right\}, \text{ and}$$

$$\frac{x}{c} = \left\{\frac{1}{1-2\frac{y^2}{x^2}}\right\}^{\frac{3}{4}}, \text{ or } (x^2-2y^2)^3 = Cx^2.$$

3. Integrate $x\,dx + y\,dy = x\,dy - ydx$.

Assuming $y = vx$, we have

$$x\,dx + vx^2\,dv + v^2\,xdx = x^2\,dv.$$

$$\therefore \frac{dx}{x} = \frac{1-v}{1+v^2}\,dv,$$

and by integration and reduction,

$$\log\sqrt{x^2+y^2} = \text{tang}^{-1}\frac{y}{x} + C.$$

4. Integrate $x^2\,ydx - y^3\,dy = x^3\,dy$.

Let $y = vx$; then we shall have, by substitution and reduction,

$$\frac{dv}{v} + \frac{dx}{x} = -\frac{dv}{v^4}.$$

$$\therefore \log\frac{vx}{c} = \log\frac{y}{c} = \frac{1}{3}\,v^{-3} = \frac{x^3}{3y^3}.$$

$$\therefore y = ce^{\frac{x^3}{3y^3}}.$$

5. Integrate $x^3\,dy - x^2\,y\,dx + y^3\,dx - xy^2\,dy = 0$.

Assuming $y = vx$, we find

$$x\,(1-v^2)\,dv = 0 \qquad (a).$$

$$\therefore dv = 0 \text{ and } v = c; \text{ whence } y = cx.$$

The reduced equation (a) may also be solved by making $x = 0$ and $1 - v^2 = 0$, or $v = \pm 1$; whence $y = c \times 0$,

or $y = \pm x$. These values of y are evidently *particular* integrals corresponding to the values $c = \infty$ and $c = \pm 1$.

6. Integrate $xy\,dy - y^2\,dx = (x + y)^2 e^{-\frac{y}{x}}\,dx$.

Ans. $x = ce^{\frac{xe^{\frac{y}{x}}}{x+y}}$.

7. Integrate $dy + b^2 y^2\,dx = a^2 x^m\,dx$.

The integral of this, which is known as **Riccati's** equation, has been found only for certain special values of m.

(a). If $m = 0$, we have at once

$$dy + b^2 y^2\,dx = a^2\,dx; \text{ whence}$$

$$dx = \frac{dy}{a^2 - b^2 y^2}.$$

$$\therefore\ x = \int \frac{dy}{a^2 - b^2 y^2} = \frac{1}{2ab} \log c \left\{ \frac{a + by}{a - by} \right\}, \text{ or}$$

$$\frac{a + by}{a - by} = Ce^{2abx}.$$

(b). If $m = -2$, let $y = v^{-1}$. Then we shall have

$$-\frac{dv}{v^2} + b^2 \frac{dx}{v^2} = a^2 \frac{dx}{x^2}.$$

Substituting ux for v, and reducing, we have

$$\frac{dx}{x} = -\frac{du}{a^2 u^2 + u - b^2}$$

an equation which is immediately integrable by the rules for rational fractions.

(c). Assuming $y = \dfrac{1}{b^2 x} + \dfrac{v}{x^2}$, we obtain, by differentiation and reduction,

$$dv + b^2 v^2 \frac{dx}{x^2} = a^2 x^{m+2}\,dx.$$

If $m = -4$, this equation becomes

$$dv + b^2 v^2 \frac{dx}{x^2} = a^2 \frac{dx}{x^2}, \text{ or}$$

$$\frac{dx}{x^2} = \frac{dv}{a^2 - b^2 v^2},$$

from which we easily find, by integration,

$$\frac{ab + x - b^2 x^2 y}{ab - x + b^2 x^2 y} = ce^{\frac{2ab}{x}}.$$

(*d*). If in the equation

$$dv + b^2 v^2 \frac{dx}{x^2} = a^2 x^{m+2} dx,$$

we assume $v = \dfrac{1}{u}$, we shall have

$$du + a^2 u^2 x^{m+2} dx = b^2 \frac{dx}{x^2};$$

if in this equation we make $x^{m+2} dx = \dfrac{dt}{m+3}$, and put $a^2 = \beta^2 (m+3)$, $b^2 = \alpha^2 (m+3)$, $\dfrac{m+4}{m+3} = -n$, it reduces to

$$du + \beta^2 u^2 dt = \alpha^2 t^n dt.$$

This equation, being of the same form as the original, is necessarily integrable when $n = -4$ or $m = -\dfrac{8}{3}$.

(*e*). If n is not equal to -4, we may, by means of the processes employed in cases (*c*) and (*d*), reduce this last equation to a similar equation,

$$du' + \beta'^2 u'^2 dt' = \alpha'^2 t'^{n'} dt',$$

which can be integrated when $n' = -4$, whence $n = -\dfrac{8}{3}$ and $m = -\dfrac{12}{5}$.

Similarly, if n' be not equal to -4, we may convert this last equation into an equivalent one

$$du'' + \beta''^2 u''^2 dt'' = \alpha''^2 t''^{n''} dt'',$$

which is integrable when $n'' = -4$, whence $n' = -\frac{8}{3}$, $n = -\frac{12}{5}$, and $m = -\frac{16}{7}$; and so on.

The given equation is integrable, therefore, whenever the value of m is a term of the following series:

$$0, \quad -4, \quad -\frac{8}{3}, \quad -\frac{12}{5}, \quad -\frac{16}{7}, \text{ etc.};$$

all of which are included in the general form

$$m = -\frac{4r}{2r-1},$$

in which r may be any positive whole number.

(*f*). If in the original equation we make $y = \frac{1}{u}$, it becomes

$$du + a^2 u^2 x^m dx = b^2 dx;$$

and if we make $x^m dx = \frac{dv}{m+1}$; $\quad a^2 = \beta^2 (m+1)$;

$$b^2 = \alpha^2 (m+1); \quad \frac{m}{m+1} = -n,$$

we shall have $\quad du + \beta^2 u^2 dv = \alpha^2 v^n dv,$

an equation entirely similar to the original, and therefore integrable when $n = -\frac{4r}{2r-1}$; whence

$$\frac{m}{m+1} = \frac{4r}{2r-1}, \text{ and } m = \frac{-4r}{2r+1}.$$

Combining the last two cases we say, then, that Riccati's equation is integrable whenever $m = \frac{-4r}{2r \pm 1}$, and by pursuing the methods indicated in the foregoing discussion, we may reduce any particular equation to an equivalent one in which the exponent $m = -4$, or -2, or 0, which will be directly integrable by one of the first three cases.

8. $$dy + y^2\,dx = x^{-\frac{4}{3}}\,dx \qquad (1).$$

We have $m = -\frac{4}{3} = \frac{-4(1)}{2(1)+1}$. Hence we must put, according to (f), $y = \frac{1}{u}$; whence

$$du + u^2 x^{-\frac{4}{3}}\,dx = dx \qquad (2).$$

Now, making $x^{-\frac{4}{3}}dx = \dfrac{dv}{-\frac{4}{3}+1}$; $\beta^2 = \dfrac{1}{-\frac{4}{3}+1}$;

$\alpha^2 = \dfrac{1}{-\frac{4}{3}+1}$; $\dfrac{-\frac{4}{3}}{-\frac{4}{3}+1} = 4 = -n$; we have

$$du - 3u^2\,dv = -3v^{-4}\,dv.$$

Since in this equation the exponent of v is -4, we must, according to (c), put $u = -\frac{1}{3v} + \frac{z}{v^2}$; whence, by substitution,

$$\frac{3dv}{v^2} = \frac{dz}{z^2-1},$$

and by integration,

$$-\frac{6}{v} = \log\left\{\frac{z-1}{z+1}\right\}, \quad \text{or} \quad \frac{z-1}{z+1} = ce^{-\frac{6}{v}}.$$

Substituting in this last equation the values of z and v, we have

$$ce^{-6x^{\frac{1}{3}}} = \frac{3v^2 u + v - 3}{3v^2 u + v + 3} = \frac{3x^{-\frac{2}{3}} y^{-1} + x^{-\frac{1}{3}} - 3}{3x^{-\frac{2}{3}} y^{-1} + x^{-\frac{1}{3}} + 3}$$

$$= \frac{3 + yx^{\frac{1}{3}} (1 - 3x^{\frac{1}{3}})}{3 + yx^{\frac{1}{3}} (1 + 3x^{\frac{1}{3}})};$$

or, finally, $$Ce^{6x^{\frac{1}{3}}} = \frac{3 + yx^{\frac{1}{3}} (1 + 3x^{\frac{1}{3}})}{3 + yx^{\frac{1}{3}} (1 - 3x^{\frac{1}{3}})},$$

which is the complete integral of the given equation.

9. $$dy - y^2\,dx = 2x^{-\frac{8}{3}}\,dx \qquad (1).$$

Let $$y = -\frac{1}{x} + \frac{v}{x^2}.$$

Then (1) becomes

$$dv - v^2 \frac{dx}{x^2} = 2x^{-\frac{8}{3}+2}\,dx \qquad (2).$$

Let $v = \frac{1}{u}$. Then (2) becomes

$$du + 2u^2 x^{-\frac{8}{3}+2}\,dx = -\frac{dx}{x^2} \qquad (3).$$

Let $$x^{-\frac{8}{3}+2} dx = \frac{dt}{-\frac{8}{3}+3}; \text{ whence } t = x^{\frac{1}{3}}.$$

Then (3) becomes

$$du + 6u^2\,dt = -3t^{-4}\,dt \qquad (4).$$

Let $$u = \frac{1}{6t} + \frac{s}{t^2}.$$

Then (4) becomes $-\dfrac{dt}{t^2} = \dfrac{ds}{3 + 6s^2}$ (5),

by integrating which we obtain

$$c + \frac{1}{t} = \frac{1}{3\sqrt{2}} \operatorname{tang}^{-1} s\sqrt{2}; \text{ whence}$$

$$s\sqrt{2} = \operatorname{tang}\left\{\left(\frac{3\sqrt{2}}{t}\right) + C\right\} = \operatorname{tang}\left\{\left(\frac{3\sqrt{2}}{x^{\frac{1}{3}}}\right) + C\right\} \quad (6).$$

But, from the above substitutions, we have

$$s = \frac{6t^2 u - t}{6} = \frac{6t^2 - tv}{6v} = \frac{6t^2 - t(x^2 y + x)}{6(x^2 y + x)}$$

$$= \frac{6x^{\frac{2}{3}} - (x^{\frac{7}{3}} y + x^{\frac{4}{3}})}{6(x^2 y + x)}.$$

$$\therefore \; s\sqrt{2} = \frac{6 - (x^{\frac{5}{3}} y + x^{\frac{2}{3}})}{3\sqrt{2}\, x^{\frac{1}{3}} (xy + 1)} = \operatorname{tang}\left\{\frac{3\sqrt{2}}{x^{\frac{1}{3}}} + C\right\}.$$

10. Find the curve in which the subtangent at any point is equal to the sum of its coördinates.

We have, without regard to the sign of the subtangent,

$$y\frac{dx}{dy} = x + y;$$

$$\therefore \; y\,dx - x\,dy = y\,dy.$$

Making $x = yv$, we have, by differentiation and reduction,

$$\frac{dy}{y} = dv.$$

$\therefore \; \log\left(\dfrac{y}{c}\right) = v = \dfrac{x}{y}$, and $\dfrac{y}{c} = e^{\frac{x}{y}}$; whence the equation of the required curve is

$$y = ce^{\frac{x}{y}}.$$

11. The curve whose subtangent is constant.

Here $$y\frac{dx}{dy} = a, \text{ or } y\,dx = a\,dy.$$

$$\therefore \frac{dy}{y} = \frac{dx}{a}, \text{ and } \log\left(\frac{y}{c}\right) = \frac{x}{a}.$$

$$\therefore y = ce^{\frac{x}{a}},$$ the equation of the logarithmic curve.

12. The curve whose subnormal is constant.

Here $$y\frac{dy}{dx} = a, \text{ or } y\,dy = a\,dx,$$

from which we have at once

$$y^2 = 2ax + C$$ — the parabola.

13. The curve which cuts at a constant angle all lines whose equation is

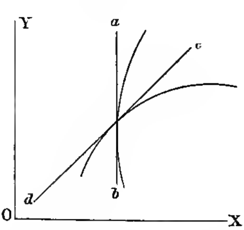
$$F(x, y, a) = 0, \tag{1}$$

in which a may have all possible values.

If we designate by m the tangent of the given angle, by $x'\,y'$ the coördinates of a point on the required curve, and by a the angle which the tangent line to the given curve makes with the axis of abscissas, we shall have

$$m = \frac{\dfrac{dy'}{dx'} - \text{tang } a}{1 + \dfrac{dy'}{dx'}\text{ tang } a} \tag{2}$$

Fig. 42.



Now the tangent of a is the value of $\dfrac{dy}{dx}$ derived from the equation (1) for the particular point of intersection $x'\,y'$.

Hence, $$\text{tang } a = -\frac{\dfrac{dF}{dx'}}{\dfrac{dF}{dy'}},$$

and by substitution in (2),

$$m\left\{\frac{dF}{dy'}-\frac{dF}{dx'}\frac{dy'}{dx'}\right\}-\frac{dF}{dy'}\frac{dy'}{dx'}-\frac{dF}{dx'}=0 \qquad (3).$$

If we eliminate a between this equation and $F(x', y', a) = 0$, the resulting equation will be a relation between x' and y', and will therefore be the equation of the required curve. This curve is called a **Trajectory.**

14. The curve which cuts at an angle of 45° all straight lines drawn through a given point.

Let the given point be the origin. Then we shall have

$$F(x, y, a) = y - ax = 0.$$

$$\therefore \frac{dF}{dy'} = 1; \quad \frac{dF}{dx'} = -a; \quad m = \text{tang } 45^\circ = 1.$$

Therefore, by substitution in (3),

$$x\,dx + y\,dy - x\,dy + y\,dx = 0,$$

a *homogeneous* equation, from which we obtain, by integration,

$$\log\frac{\sqrt{x^2+y^2}}{c} = \text{tang}^{-1}\frac{y}{x},$$

which is the equation of the required curve.

Transferring to polar coördinates by means of the equations $y = r \sin\theta$, $x = r\cos\theta$, whence

$$\text{tang }\theta = \frac{y}{x}, \text{ and } \sqrt{x^2+y^2} = r, \text{ we have}$$

$$\log\frac{r}{c} = \theta, \text{ or}$$

$$r = ce^{\theta} \text{ — the logarithmic spiral.}$$

15. The curve which cuts at right angles all parabolas with coincident vertices and axes.

Here $$F(x, y, a) = y^2 - 2px = 0;$$

$$\frac{dF}{dy'} = 2y; \quad \frac{dF}{dx'} = -2p = -\frac{y^2}{x}; \quad m = \infty.$$

The value of m being infinite, the coëfficient of m in (3) must be zero.

$$\therefore \quad 2y + \frac{y^2}{x}\frac{dy}{dx} = 0, \text{ or } 2x\,dx + y\,dy = 0;$$

from which we obtain, by integration,

$$x^2 + \frac{1}{2}y^2 = c^2 \text{ —— an ellipse.}$$

CHAPTER XIII.

INTEGRATION OF EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREES.

83. The number of general forms of these equations which have been integrated is comparatively limited, and we shall confine ourselves to the examination of a few of the more important cases.

84. Case I.—When an equation can be solved with reference to $\frac{dy}{dx}$, each resulting value of $\frac{dy}{dx}$ will give us a new equation of the form $Pdx + Qdy = 0$, which can be integrated by the methods already established.

Let $\phi(x, y, c) = 0$, $\phi(x, y, c') = 0$, $\phi(x, y, c'') = 0 \ldots$, be the various integrals obtained in this manner. Then the equation

$$\{\phi(x, y, c)\}\{\phi(x, y, c')\}\{\phi(x, y, c'')\} \ldots = 0$$

will evidently contain each of these particular integrals, and will therefore be the complete integral of the given equation; and since each of the factors is separately equal to zero, we may take the same value for the constant in all of them.

EXAMPLES.

1. $$\left\{\frac{dy}{dx}\right\}^2 = a^2.$$

We have $\frac{dy}{dx} = \pm a$, whence

$$y = ax + c \text{ and } y = -ax + c; \text{ or}$$

$$(y - ax - c)(y + ax - c) = 0,$$

which is the complete integral of the given equation.

2. $$\left(\frac{dy}{dx}\right)^3 - 6\left(\frac{dy}{dx}\right)^2 + 11\left(\frac{dy}{dx}\right) - 6 = 0.$$

We have $\frac{dy}{dx} = 1, 2$, and 3; whence,

$y = x + c$; $y = 2x + c$; $y = 3x + c$; and the complete integral is

$$(y - x - c)(y - 2x - c)(y - 3x - c) = 0.$$

3. $$a^2\left(\frac{dy}{dx}\right)^3 + a^2bx\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} = x^2\left(\frac{dy}{dx}\right)^3 + bx^3\left(\frac{dy}{dx}\right)^2 + bx.$$

Putting this under the form

$$\left\{\frac{dy}{dx} + bx\right\}\left\{\frac{dy}{dx} + \frac{1}{\sqrt{a^2 - x^2}}\right\}\left\{\frac{dy}{dx} - \frac{1}{\sqrt{a^2 - x^2}}\right\} = 0,$$

we easily obtain the integrals. The complete integral is

$$(y - c)^3 = \frac{bx^2}{2}\left(\sin^{-1}\frac{x}{a}\right)^2.$$

85. If we have, generally, $F\left(\frac{dy}{dx}\right) = 0$, and can find a, a root of the similar finite equation $F(z) = 0$, we may evidently satisfy the given equation by taking $a = \frac{dy}{dx}$; whence $y = ax + c$, and $a = \frac{y-c}{x}$. We may then write as the complete integral of the given equation

$$F\left(\frac{y-c}{x}\right) = 0.$$

EXAMPLE.

To find the curve which has the property $s = ax + by$.

We have $\frac{ds}{dx} = a + b\frac{dy}{dx}$; whence

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = a + b\frac{dy}{dx}.$$

$\therefore$ the required equation is

$$\sqrt{1 + \left(\frac{y-c}{x}\right)^2} = a + b\frac{y-c}{x}.$$

This equation may be resolved into two factors of the first degree, each of which, placed equal to zero, is the equation of a straight line.

86. Case II.—Let the form be the homogeneous equation

$$\left(\frac{dy}{dx}\right)^m + F\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)^{m-1} + \cdots + f\left(\frac{y}{x}\right) = 0.$$

If we put $\frac{y}{x} = u$, whence $y = ux$, and $\frac{dy}{dx} = u + x\frac{du}{dx}$, the equation becomes

$$\left\{u + x\frac{du}{dx}\right\}^m + F(u)\left\{u + x\frac{du}{dx}\right\}^{m-1} + \cdots + f(u) = 0.$$

Solving this equation, if possible, we shall have m values of $u + x\frac{du}{dx}$, each of which may be put under the form

$$u + x\frac{du}{dx} = \phi(u), \text{ or } \frac{dx}{x} = \frac{du}{\phi(u) - u},$$

in which the variables are separated.

EXAMPLE.

$$y - x\frac{dy}{dx} = x\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

Assuming $\frac{y}{x} = u$, we have

$$u - \frac{dy}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

$$\therefore\ u - \left(u + x\frac{du}{dx}\right) = \left\{1 + \left(u + x\frac{du}{dx}\right)^2\right\}^{\frac{1}{2}};$$

$$x^2\left(\frac{du}{dx}\right)^2 = 1 + u^2 + 2ux\frac{du}{dx} + x^2\left(\frac{du}{dx}\right)^2;$$

$$1 + u^2 + 2ux\frac{du}{dx} = 0;$$

$$\frac{dx}{x} = -\frac{2u\,du}{1 + u^2};$$

$$\log\frac{x}{c} = -\log(1 + u^2) = \log\frac{1}{1 + u^2} = \log\frac{x^2}{x^2 + y^2};$$

$$\frac{x}{c} = \frac{x^2}{x^2 + y^2}, \text{ and } cx = x^2 + y^2.$$

87. Case III.—When the given equation can not be resolved with respect to $\frac{dy}{dx}$ it may be integrated provided we can solve it with respect to one of the variables.

Let the resolved equation be of the form

$$y = xF\left(\frac{dy}{dx}\right) + f\left(\frac{dy}{dx}\right).$$

Representing $\frac{dy}{dx}$ by p, and differentiating, we obtain

$$pdx = F(p)\,dx + xF'(p)\,dp + f'(p)\,dp, \text{ or}$$

$$dx + \frac{x}{F(p) - p}\,F'(p)\,dp = -\frac{f'(p)\,dp}{F(p) - p}.$$

This is a *linear* equation [Art. 80], and will give, by integration, a relation between x and p. Eliminating p between this and the given equation, we shall have a finite relation between x and y.

Examples.

1. $$y = (1 + p)\,x + p^2.$$

We have $F(p) = 1 + p;\ f(p) = p^2;\ F(p) - p = 1;$

$$F'(p)\,dp = dp;\ f'(p)\,dp = 2pdp.$$

$$\therefore\ dx + x\,dp = -2pdp,$$

the integral of which is

$$x = e^{-\int dp}\left\{C - 2\int e^{\int dp}\,pdp\right\} = Ce^{-p} + 2(1 - p).$$

Substituting in this the value of p derived from the given equation, we have

$$Ce^{\frac{x}{2} \mp \frac{1}{2}\sqrt{4y - 4x + x^2}} \mp \sqrt{4y - 4x + x^2} + 2 = 0.$$

2. $$y = 2px + \sqrt{1 + p^2}.$$

We have $F(p) = 2p$; $f(p) = \sqrt{1+p^2}$; $F(p) - p = p$;

$$F'(p)\,dp = 2dp;\ f'(p)\,dp = (1+p^2)^{-\frac{1}{2}}\,pdp.$$

$$\therefore\ dx + \frac{2x\,dp}{p} + \frac{dp}{\sqrt{1+p^2}} = 0,$$

the integral of which is

$$x = e^{-\int\frac{2dp}{p}}\left\{C - \int e^{\int\frac{2dp}{p}}\frac{dp}{\sqrt{1+p^2}}\right\}$$

$$= e^{\log\frac{1}{p^2}}\left\{C - \int e^{\log p^2}\frac{dp}{\sqrt{1+p^2}}\right\}$$

$$= \frac{1}{p^2}\left\{C - \int\frac{p^2\,dp}{\sqrt{1+p^2}}\right\}.$$

$$\therefore\ p^2x = C - \frac{1}{2}(1+p^2)^{\frac{1}{2}}p + \frac{1}{2}\log\{p + (1+p^2)^{\frac{1}{2}}\}.$$

The elimination of p between this and the given equation will give us a finite relation between x and y.

88. Case IV.—Let the resolved equation be

$$y = px + f(p),$$

which is known as **Clairault's** form.

Differentiating with respect to p, we have

$$0 = \{x + f'(p)\}\,dp \qquad (a).$$

This equation gives $dp = 0$; whence $p = c$, and therefore

$$y = cx + f(c).$$

The given equation may therefore be integrated by replacing p by an arbitrary constant.

Equation (a) also gives us $x + f'(p) = 0$. If we eliminate p between this result and the given equation, we shall obtain an integral without an arbitrary constant, and therefore not included in the general integral; *i. e.*, it will be what we have called a singular solution.

EXAMPLES.

1. $$y = px + n\sqrt{1 + p^2}.$$

Here $$f(p) = \sqrt{1 + p^2}.$$

$\therefore$ the general integral is

$$y = cx + n\sqrt{1 + c^2}.$$

2. $$y = px - ap(1 + p^2)^{-\frac{1}{2}}.$$

The general integral is

$$y = cx - ac(1 + c^2)^{-\frac{1}{2}}.$$

3. $$ay\left(\frac{dy}{dx}\right)^2 + (2x - b)\frac{dy}{dx} - y = 0.$$

Assuming $y^2 = u$, whence $2y\,dy = du$, we have

$$a\left(\frac{du}{dx}\right)^2 + (4x - 2b)\frac{du}{dx} - 4u = 0.$$

$$\therefore \quad u = x\frac{du}{dx} - \frac{b}{2}\frac{du}{dx} + \frac{a}{4}\left(\frac{du}{dx}\right)^2,$$

which is of Clairault's form.

The general integral is

$$u = y^2 = cx - \frac{bc}{2} + \frac{ac^2}{4}.$$

4. $$axy\left(\frac{dy}{dx}\right)^2 + (bx^2 - ay^2 - ab)\frac{dy}{dx} = bxy.$$

Assuming $x^2 = v$, and $y^2 = u$, this equation reduces to

$$u = v\frac{du}{dv} - \frac{ab\frac{du}{dv}}{b + a\frac{du}{dv}}$$

which is of Clairault's form.

Therefore the general integral is

$$y^2 = cx^2 - \frac{abc}{b + ac}.$$

CHAPTER XIV.

SINGULAR SOLUTIONS OF EQUATIONS OF THE FIRST ORDER.

89. Let $$F(x, y, a) = 0 \qquad (1)$$

be the general integral of a differential equation of the first order; a being the constant which by elimination between (1) and its first derivative

$$\frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0 \qquad (2)$$

has led to the given differential equation.

We remark that in every equation of the form (1), we may replace a by ϕ, a function of x and y.

For, assuming $F(x, y, \phi) = F(x, y, a)$,

we may determine ϕ in such a manner as to render this an identical equation; and consequently

$$F(x, y, \phi) = 0 \qquad (3),$$

may be considered as an integral of the given equation.

This integral, not being included in (1), is what we have called a *singular* integral, and it remains now to determine ϕ.

The given equation is, as we have seen, the result of the elimination of a between (1) and its first derivative, and since a may be replaced by ϕ, it follows that the given equation is also the result of the elimination of ϕ between $F(x, y, \phi) = 0$ and its first derivative.

Now, the first derivative of (3) is

$$\frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx} + \frac{dF}{d\phi}\frac{d\phi}{dx} = 0 \qquad (4);$$

and in order that the elimination of ϕ may produce the same result as that of a between (1) and (2), we must evidently have

$$\frac{\frac{dF}{d\phi}\frac{d\phi}{dx}}{\frac{dF}{dy}} = 0 \qquad (5).$$

This equation may be satisfied in two ways.

1*st.* If we put $\frac{d\phi}{dx} = 0$, we obtain $\phi = c$, and the result of substituting this value of ϕ in (3) will be the *general* integral.

2*d.* If we put $\frac{dF}{d\phi} = 0$, and substitute the resulting value of ϕ in (3), the equation thus obtained will be a singular solution if it is not included in (1). This result will be indentical with that obtained by eliminating a between the equations $F(x, y, a) = 0$ and $\frac{dF}{da} = 0$; and we have therefore the following

Rule for obtaining Singular Solutions.—*Find the general integral of the given equation, which will necessarily involve one arbitrary constant. Differentiate the integral with respect to this constant, and eliminate the constant between the integral and its derivative.*

The equation thus obtained will be a singular solution, *provided it is not itself included in the general integral.*

90. It is evident that (5) may be satisfied by placing $\frac{dF}{dy} = \infty$, and hence we may also find a singular solution by eliminating the constant between the equations

$$F(x, y, a) = 0, \text{ and } \frac{dF}{dy} = \infty.$$

But in all cases it is necessary to observe whether either or both of these methods will not reduce (5) to the form $\frac{0}{0}$. If this should be the case, there would be no singular solution unless the real value of $\frac{0}{0}$ should be *zero.*

91. It will be observed that the rule above given is identical with that for finding the envelope of a series of curves [Diff. Cal., Art. 136]. Whence it follows that *the curve, of which the singular solution is the equation, is the envelope of the series of curves given by causing the constant in the general integral to vary continuously.*

92. The foregoing consideration will lead to the following simple method of determining, in many cases, the singular solution without first obtaining the general integral.

Let the differential equation be written

$$f(x, y, p) = 0 \qquad (6),$$

in which $p = \frac{dy}{dx}$.

Now, since (6) is the differential equation of both the general and the singular solutions, and since the envelope is, in general, tangent to each of the curves of which the general integral is the equation, it follows that for any values of x and y which satisfy both the general and the singular solutions, (6) ought to give *two equal values* of p or $\frac{dy}{dx}$, one of

which is the tangent of the angle which the tangent line to the envelope makes with the axis of x, and the other is the tangent of the angle which the tangent line to the given curve makes with the same axis.

If, therefore, $$\phi(x, y, p) = 0 \tag{7}$$

be an equation which expresses the condition that (6) shall give two equal values of p for the same values of x and y, the singular solution ought to satisfy both (6) and (7), and therefore the equation resulting from the elimination of p between them. If, then, we effect this elimination, and the resulting equation satisfies (6), it will be the singular solution.

93. If in (6) the function f be of such a form that it can have but one value for a given value of p, we may readily find equation (7).

In fact, it is shown in algebra that when an equation has two equal roots, there is a factor common to itself and its first derived polynomial (which is just what we have called in the Calculus the first derivative); consequently, in the case under consideration, there must exist a value of p common to the two equations (6) and $\dfrac{df}{dp} = 0$.

Hence, in all cases where p is not involved in *radical* terms, we may find the singular solution by eliminating p between the two equations $f(x, y, p) = 0$ and $\dfrac{df}{dp} = 0$, the latter being the form assumed by (7) in this case.

94. EXAMPLES.

1. $$y = xp + a\sqrt{1 + p^2}.$$

This being of Clairault's form, the general integral is

$$y = cx + a\sqrt{1 + c^2}.$$

$$\therefore \frac{dy}{dc} = x + ac(1 + c^2)^{-\frac{1}{2}} = 0.$$

Eliminating c between these two equations, we have

$$x^2 + y^2 = a^2,$$

which is the singular solution.

2. $$y = xp - \frac{ap}{\sqrt{1+p^2}}.$$

The general integral is

$$y = cx - ac\,(1+c^2)^{-\frac{1}{2}}.$$

$$\therefore\ \frac{dy}{dc} = x - a\,(1+c^2)^{-\frac{1}{2}} + ac^2(1+c^2)^{-\frac{3}{2}} = 0.$$

Eliminating c, we have for the singular solution,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

3. $$f = y + (y-x)\,p + (a-x)\,p^2 = 0.$$

We have $\dfrac{df}{dp} = y - x + 2\,(a-x)\,p = 0.$

Eliminating p, we have for the singular solution,

$$(x+y)^2 - 4ay = 0.$$

4. $$f = ayp^2 + (2x-b)\,p - y = 0.$$

$$\therefore\ \frac{df}{dp} = 2ayp + (2x-b) = 0;$$

and the singular solution is

$$4ay^2 + (2x-b)^2 = 0.$$

5. $$f = p^2x - py + m = 0.$$

Ans. $y^2 - 4mx = 0.$

6. $$f = y^2 - 2xyp + (1+x^2)\,p^2 - 1 = 0.$$

Ans. $y^2 = 1 + x^2.$

7. $$f = (px-y)\,(px-2y) + x^3 = 0.$$

Ans. $y^2 - 4x^3 = 0.$

95. Miscellaneous Examples in Equations of the First Order.

1. $$xdy - ydx = dx\sqrt{x^2 + y^2}.$$

Let $x = vy$. Then $dx = vdy + ydv$, and

$$\sqrt{x^2 + y^2} = y\sqrt{1 + v^2}.$$

Hence $vy\,dy - vy\,dy - y^2\,dv = (vdy + ydv)\,y\sqrt{1 + v^2}$, and

$$\frac{dy}{y} = -\frac{dv}{v\sqrt{1+v^2}} - \frac{dv}{v}.$$

$$\therefore \log\frac{y}{c} = -\frac{1}{2}\log\frac{\sqrt{1+v^2}-1}{\sqrt{1+v^2}+1} - \log v$$

$$= -\log\frac{\sqrt{1+v^2}-1}{v} - \log v$$

$$= \log\frac{1}{\sqrt{1+v^2}-1}$$

$$= \log\frac{y}{\sqrt{x^2+y^2}-y}.$$

$$\therefore \frac{y}{c} = \frac{y}{\sqrt{x^2+y^2}-y},$$ or $\sqrt{x^2+y^2} = c + y$, and

$$x = \sqrt{c^2 + 2cy}.$$

2. $$ydy = (xdy + ydx)\sqrt{1 + y^2}.$$

We have $$\frac{ydy}{\sqrt{1+y^2}} = ydx + xdy.$$

$$\therefore \sqrt{1+y^2} = yx + c;$$ or

$$x = \frac{\sqrt{1+y^2}}{y} - \frac{c}{y}.$$

3. $$y^2\,dy = 3xydx - x^2\,dy.$$

Let $x = vy$. Then, by substitution and reduction, we have

$$\frac{dy}{y} = \frac{3vdv}{1-2v^2}.$$

$$\therefore\ \log\frac{y}{c} = -\frac{3}{4}\log(1-2v^2),\ \text{ and}$$

$$y = \frac{c}{(1-2v^2)^{\frac{3}{4}}} = \frac{cy^{\frac{3}{2}}}{(y^2-2x^2)^{\frac{3}{4}}}.$$

$$\therefore\ y^2 = c^{\frac{4}{3}}\,y^{\frac{2}{3}} + 2x^2.$$

4. $$\frac{xdy - ydx}{x^2+y^2} = ad\theta.$$

We have $$\frac{xdy-ydx}{x^2+y^2} = \frac{d\,\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} = d\,\text{tang}^{-1}\frac{y}{x};$$

$$ad\theta = d\,(a\theta).$$

$$\therefore\ d\,\text{tang}^{-1}\frac{y}{x} = d\,(a\theta),\ \text{ and}$$

$$\text{tang}^{-1}\frac{y}{x} = a\theta + c;\ \text{ or}$$

$$\frac{y}{x} = \text{tang}\,(a\theta + c),\ \text{ and}$$

$$y = x\,\text{tang}\,(a\theta + c).$$

5. $$xdx + ady = b\sqrt{dx^2+dy^2}.$$

We have $$x + a\frac{dy}{dx} = b\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

$$\therefore\ x^2 + 2ax\frac{dy}{dx} + a^2\left(\frac{dy}{dx}\right)^2 = b^2 + b^2\left(\frac{dy}{dx}\right)^2.$$

Solving this equation with reference to $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{ax}{b^2-a^2}\pm\frac{b}{b^2-a^2}\sqrt{x^2+a^2-b^2}.$$

$$\therefore\ dy=\frac{ax\,dx}{b^2-a^2}\pm\frac{b}{b^2-a^2}\,dx\sqrt{x^2+a^2-b^2}, \text{ and}$$

$$y=\frac{ax^2}{2(b^2-a^2)}\pm\frac{b}{b^2-a^2}\int dx\sqrt{x^2+a^2-b^2}.$$

The integral of the last term is

$$\frac{x}{2}\sqrt{x^2+a^2-b^2}+\frac{a^2-b^2}{2}\log\left(x+\sqrt{x^2+a^2-b^2}\right).$$

$$\therefore\ y=\frac{ax^2}{2(b^2-a^2)}\pm\frac{bx\sqrt{x^2+a^2-b^2}}{2(b^2-a^2)}$$

$$\mp\frac{b}{2}\log\left(x+\sqrt{x^2+a^2-b^2}\right)+C.$$

6. $$xdy-ydx=\frac{2xdy-ydx}{\sqrt{x^2+y^2}}.$$

Substituting vy for x, we have by differentiation and reduction,

$$\frac{ydv-vdy}{y^2}=dv\sqrt{1+v^2}.$$

$$\therefore\ \frac{v}{y}=\int dv\sqrt{1+v^2}$$

$$=\frac{1}{2}v\sqrt{1+v^2}+\frac{1}{2}\log\left(v+\sqrt{1+v^2}\right)+C; \text{ or}$$

$$\frac{x}{y^2}=\frac{1}{2}\,\frac{x}{y}\sqrt{1+\frac{x^2}{y^2}}+\frac{1}{2}\log\left(\frac{x}{y}+\sqrt{1+\frac{x^2}{y^2}}\right)+C.$$

$$\therefore\ x=\frac{1}{2}x\sqrt{x^2+y^2}+\frac{1}{2}y^2\log\frac{x+\sqrt{x^2+y^2}}{y}+Cy^2.$$

7. $$\frac{dv}{\sqrt{Av^2 + Bv + C}} + \frac{du}{\sqrt{Au^2 + Bu + C}} = 0.$$

Assume $B = 2aA$; $C = Ab$; $v = x - a$; $u = y - a$.

Then we have by substitution and reduction,

$$\frac{dx}{\sqrt{x^2 + b - a^2}} + \frac{dy}{\sqrt{y^2 + b - a^2}} = 0.$$

$$\therefore\ \log\{x + \sqrt{x^2 + b - a^2}\} + \log\{y + \sqrt{y^2 + b - a^2}\} = \log c,$$

and $\{x + \sqrt{x^2 + b - a^2}\}\{y + \sqrt{y^2 + b - a^2}\} = c$; or

$$\left\{v + \frac{B}{2A} + \sqrt{v^2 + \frac{B}{A}v + \frac{C}{A}}\right\}\left\{u + \frac{B}{2A} + \sqrt{u^2 + \frac{B}{A}u + \frac{C}{A}}\right\} = c.$$

8. $$dx + x^3\,dx = du + u\,dx$$

Assume $u = yv$. Then we shall have

$$(1 + x^3)\,dx = y\,dv + v\,dy + yv\,dx.$$

The quantities y and v being arbitrary, we may assume

$v\,dy + yv\,dx = 0$; whence $dy + y\,dx = 0$, and

$\frac{dy}{y} = -\,dx$, or $\log\frac{y}{c} = -\,x$, and $y = ce^{-x}$.

Also $(1 + x^3)\,dx = y\,dv$; whence

$$dv = \frac{(1 + x^3)\,dx}{y} = \frac{(1 + x^3)\,dx}{ce^{-x}}$$

$$= \frac{1}{c}\,e^x\,dx + \frac{1}{c}\,e^x\,x^3\,dx.$$

$$\therefore \; v = \frac{1}{c} e^x + \frac{1}{c} \int e^x x^3 \, dx$$

$$= \frac{1}{c} e^x + \frac{1}{c} \{e^x x^3 - 3e^x x^2 + 6xe^x - 6e^x + c'\}$$

$$= \frac{1}{c} e^x \{x^3 - 3x^2 + 6x - 5\} + \frac{c'}{c}.$$

$$\therefore \; u = yv = ce^{-x} \left\{ \frac{1}{c} e^x (x^3 - 3x^2 + 6x - 5) \right\} + c' e^{-x}$$

$$= x^3 - 3x^2 + 6x - 5 + \frac{c'}{e^x}.$$

9. $$\left(\frac{dy}{dx}\right)^3 - (x^2 + xy + y^2) \left(\frac{dy}{dx}\right)^2$$

$$+ xy (x^2 + xy + y^2) \frac{dy}{dx} - x^3 y^3 = 0.$$

This may be resolved into the three equations

$$\frac{dy}{dx} - x^2 = 0; \quad \frac{dy}{dx} - xy = 0; \quad \frac{dy}{dx} - y^2 = 0;$$

the integrals of which are

$$y = \frac{x^3}{3} + c; \quad y = e^{\frac{1}{2} x^2} + c'; \quad y = -\frac{1}{x} + c''.$$

The general integral is

$$\left(y - \frac{x^3}{3} - c\right) \left(y - e^{\frac{1}{2} x^2} - c'\right) \left(y + \frac{1}{x} - c''\right) = 0.$$

CHAPTER XV.

EQUATIONS OF THE SECOND AND HIGHER ORDERS.

96. There is no general method of integrating equations of an order superior to the first. The usual mode of procedure is to make such transformations upon the given equation as will reduce the order of the derivatives contained therein. We shall present a few instances in which this method may be successfully adopted, beginning with equations of the second order, the general form of which is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

97. Case I. — Let the form be

$$F\left(x, \frac{d^2y}{dx^2}\right) = 0.$$

Resolving this equation with reference to $\frac{d^2y}{dx^2}$, we have

$$\frac{d^2y}{dx^2} = F_1(x) = X_1.$$

Multiplying by dx, and integrating, we have

$$\int \frac{d^2y}{dx^2}\,dx = \frac{dy}{dx} = \int X_1\,dx = X_2 + C_1.$$

Multiplying again by dx, and integrating, we have

$$y = \int X_2\,dx + \int C_1\,dx = X_3 + C_1 x + C_2.$$

EXAMPLE. $$\frac{d^2y}{dx^2} = ax^4.$$

We have $$\frac{d^2y}{dx^2}\,dx = ax^4\,dx.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{5}ax^5 + C_1; \quad dy = \frac{1}{5}ax^5\,dx + C_1\,dx;$$

$$y = \frac{1}{5.\,6}ax^6 + C_1x + C_2.$$

98. Case II.—Let the form be

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

If in this equation we put $\frac{dy}{dx} = p$, whence $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, we shall have

$$F\left(x, p, \frac{dp}{dx}\right) = 0,$$

an equation of the first order with respect to p.

Solving this equation, if possible, we shall obtain the value of p or $\frac{dy}{dx}$, which may then be integrated by the methods heretofore established.

Example.

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{a^2}{2x}\frac{d^2y}{dx^2}.$$

Assuming $\frac{dy}{dx} = p$, we have

$$(1+p^2)^{\frac{3}{2}} = \frac{a^2}{2x}\frac{dp}{dx}, \text{ whence}$$

$$\frac{dp}{(1+p^2)^{\frac{3}{2}}} = \frac{2x\,dx}{a^2}.$$

$$\therefore \quad \frac{p}{\sqrt{1+p^2}} = \frac{x^2}{a^2} + c = \frac{x^2+ab}{a^2}.$$

From this equation we obtain

$$p = \frac{dy}{dx} = \frac{x^2 + ab}{\{a^4 - (x^2 + ab)^2\}^{\frac{1}{2}}}, \text{ or}$$

$$dy = \frac{(x^2 + ab)\,dx}{\{a^4 - (x^2 + ab)^2\}^{\frac{1}{2}}}, \text{ and}$$

$$y = \int \frac{(x^2 + ab)\,dx}{\{a^4 - (x^2 + ab)^2\}^{\frac{1}{2}}},$$

which is the equation of the *elastic* curve.

99. Case III.—Let the form be

$$F\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

We have at once

$$F\left(p, \frac{dp}{dx}\right) = 0,$$

an equation of the first order.

EXAMPLE. $$\frac{d^2 y}{dx^2} = a + b\left(\frac{dy}{dx}\right)^2.$$

We have $$\frac{dp}{dx} = a + bp^2; \text{ whence}$$

$$dx = \frac{dp}{a + bp^2}, \text{ and}$$

$$x = \frac{1}{\sqrt{ab}} \operatorname{tang}^{-1} \sqrt{\frac{b}{a}}\, p + c, \text{ or}$$

$$p = \sqrt{\frac{a}{b}} \operatorname{tang} (x\sqrt{ab} - c\sqrt{ab}).$$

Also, $$\frac{dy}{dx} = p; \quad dy = pdx = \frac{pdp}{a + bp^2}.$$

$$\therefore\ y = \frac{1}{2b} \log \{a + bp^2\} + c'; \text{ and}$$

$$y = \frac{1}{2b} \log \{a + a \operatorname{tang}^2 (x - c) \sqrt{ab}\} + c'.$$

100. Case IV.—Let the form be

$$F\left(y, \frac{d^2 y}{dx^2}\right) = 0.$$

In this case we may, if we choose, change the independent variable, which process will convert the equation into one of the form

$$F\left(y, \frac{dx}{dy}, \frac{d^2 x}{dy^2}\right) = 0,$$

the integral of which may be found as in Case II; or we may proceed as follows:

Assuming $\frac{dy}{dx} = p$, we have

$$\frac{d^2 y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy}.$$

Hence, by substitution in the given equation,

$$F\left(y, p, \frac{dp}{dy}\right) = 0,$$

which is of the first order.

Example.

$$\frac{d^2 y}{dx^2} + a^2 y = 0.$$

Assuming $\frac{dy}{dx} = p$, this equation becomes

$$p\,\frac{dp}{dy} + a^2 y = 0; \text{ whence, by integration,}$$

$$\frac{p^2}{2} + \frac{a^2 y^2}{2} - c = 0.$$

$$\therefore \frac{dy^2}{dx^2} = 2c - a^2 y^2, \text{ and } dx^2 = \frac{dy^2}{2c - a^2 y^2}, \text{ or}$$

$$dx = \frac{dy}{\sqrt{2c - a^2 y^2}}.$$

$$\therefore x = \int \frac{dy}{\sqrt{2c - a^2 y^2}} = \frac{1}{a} \sin^{-1} \frac{ay}{\sqrt{2c}} + c'.$$

101. Case V.—If the form be

$$F\left(y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}\right) = 0,$$

the methods of the last case will still be applicable.

EXAMPLE.

$$\left(\frac{dy}{dx}\right)^2 - y\frac{d^2 y}{dx^2} = n\left\{\left(\frac{dy}{dx}\right)^2 + a^2\left(\frac{d^2 y}{dx^2}\right)^2\right\}^{\frac{1}{2}}.$$

This reduces to

$$p - y\frac{dp}{dy} = n\left\{1 + a^2\left(\frac{dp}{dy}\right)^2\right\}^{\frac{1}{2}},$$

which is of Clairault's form. The integral is therefore

$$p = cy + n\sqrt{1 + a^2 c^2}.$$

$$\therefore \frac{dy}{dx} = cy + n\sqrt{1 + a^2 c^2}, \text{ and}$$

$$x = \frac{1}{c} \log \{cy + n\sqrt{1 + a^2 c^2}\} + c'.$$

102. Case VI.—Let the given equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

be a homogeneous function of each of the variables.

Assuming $\frac{dy}{dx} = yt$, we shall have

$$\frac{d^2y}{dx^2} = y\frac{dt}{dx} + yt^2,$$

and dividing by y^m, the equation will become one of the first order between x and t. It will however be difficult, in most cases, to integrate the resulting equation, and therefore the following method is generally to be preferred.

Let $\quad y = ux; \quad \frac{dy}{dx} = p; \quad \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{v}{x}.$

Then we shall have

$$dy = pdx = udx + xdu, \text{ and } \frac{dx}{x} = \frac{du}{p-u}.$$

But, since $\frac{dp}{dx} = \frac{v}{x}$, we have $vdx = xdp$; or $\frac{dx}{x} = \frac{dp}{v}$.

$$\therefore \frac{du}{p-u} = \frac{dp}{v}, \text{ and } v\,du = dp\,(p-u).$$

Substituting in this last equation the value of v, taken from the given equation, we shall have an equation of the first order, from which we can find p in terms of u. Then, by means of the equation $\frac{dx}{x} = \frac{du}{p-u}$ we can find x in terms of u or $\frac{y}{x}$.

EXAMPLE.

$$x^3\frac{d^2y}{dx^2} = \left(y - x\frac{dy}{dx}\right)^2.$$

Making the substitutions indicated above, we have

$v = (u - p)^2$, and therefore from the equation

$vdu = dp\,(p - u)$ we have

$dp = (p - u)\,du.$

This being a linear equation, its integral is

$$p = u + 1 + ce^u.$$

Substituting in the equation $\dfrac{dx}{x} = \dfrac{du}{p - u}$, we have

$$\frac{dx}{x} = \frac{du}{1 + ce^u} = \frac{e^{-u}\,du}{c + e^{-u}}.$$

$\therefore\ \log \dfrac{x}{c'} = -\log\,(c + e^{-u})$; whence

$e^{-u} = \dfrac{c' - cx}{x}$, or by substituting the value of u,

$$y = -x \log \left\{ \frac{c' - cx}{x} \right\}.$$

This case applies not only to those equations which are essentially homogeneous, but also to those which can be made so by supposing x and y to be of the dimension *unity*, $\dfrac{dy}{dx}$ to be of the dimension 0, and $\dfrac{d^2 y}{dx^2}$ to be of the dimension -1. The above example is of the latter character.

103. Equations of the Higher Orders.—By the methods illustrated above, the order of any equation may be diminished by unity, and the successive application of these methods will finally reduce the equation to one of the first order. Usually, however, the resulting equation will not be integrable. We shall present two cases in which the integration is successful.

Case I.—Let the form be

$$F\left(\frac{d^{n-1} y}{dx^{n-1}},\ \frac{d^n y}{dx^n} \right) = 0.$$

If we assume $\dfrac{d^{n-1}y}{dx^{n-1}} = p$, we shall have $\dfrac{d^n y}{dx^n} = \dfrac{dp}{dx}$, and the given equation becomes

$$F\left(p, \frac{dp}{dx}\right) = 0,$$

which is of the first order.

EXAMPLE.

$$\frac{d^3 y}{dx^3} = a\,\frac{d^2 y}{dx^2}.$$

We have $\dfrac{dp}{dx} = ap$; whence $\dfrac{dp}{p} = adx$, and

$$\log \frac{p}{c} = ax;$$

$$\therefore\ p = \frac{d^2 y}{dx^2} = ce^{ax}.$$

Now put $\dfrac{dy}{dx} = u$. Then $\dfrac{d^2 y}{dx^2} = \dfrac{du}{dx}$, and therefore

$$du = ce^{ax}\,dx = \frac{c}{a}\,e^{ax}\,d\,(ax).$$

$$\therefore\ u = \frac{c}{a}\,e^{ax} + c'.$$

$$\therefore\ dy = \frac{c}{a}\,e^{ax}\,dx + c'\,dx; \text{ and}$$

$$y = \frac{c}{a^2}\,e^{ax} + c'x + c''.$$

104. Case II.—Let the form be

$$F\left(\frac{d^{n-2}y}{dx^{n-2}},\ \frac{d^n y}{dx^n}\right) = 0.$$

Assuming $\frac{d^{n-2}y}{dx^{n-2}} = p$, we have $\frac{d^{n-1}y}{dx^{n-1}} = \frac{dp}{dx}$, and $\frac{d^n y}{dx^n} = \frac{d^2 p}{dx^2}$. Therefore the equation becomes

$$F\left(p, \frac{d^2 p}{dx^2}\right) = 0,$$

which is of the second order.

EXAMPLE.

$$\frac{d^4 y}{dx^4} = \frac{d^2 y}{dx^2}.$$

We have
$$\frac{d^2 p}{dx^2} = p.$$

Multiplying this equation by $\frac{dp}{dx}\,dx$, we have

$$\frac{dp}{dx}\,\frac{d^2 p}{dx^2}\,dx = p\,\frac{dp}{dx}\,dx, \text{ or}$$

$$\frac{dp}{dx}\,d\left(\frac{dp}{dx}\right) = pdp.$$

$$\therefore \left(\frac{dp}{dx}\right)^2 = p^2 + c; \text{ and } dx = \frac{dp}{\sqrt{p^2+c}};$$

$$\therefore \quad x = \log\left\{\frac{p + \sqrt{p^2+c}}{c_1}\right\} \qquad (1).$$

Now we have

$$\frac{dy}{dx} = \int \frac{d^2 y}{dx^2}\,dx = \int p\,dx = \int \frac{p\,dp}{\sqrt{p^2+c}} = \sqrt{p^2+c} + c_2,$$

and $y = \int \frac{dy}{dx}\,dx = \int \{\sqrt{p^2+c} + c_2\}\,\frac{dp}{\sqrt{p^2+c}}$

$$= \int dp + \int c_2\,dx = p + c_2 x + c_3 \qquad (2).$$

If p be eliminated between (1) and (2), we shall have a finite relation between x and y.

From (1) we have

$$p + \sqrt{p^2 + c} = c_1 e^x; \text{ whence } p = \frac{c_1^2 e^{2x} - c}{2c_1 e^x}.$$

$$\therefore\ y = \frac{c_1^2 e^{2x} - c}{2c_1 e^x} + c_2 x + c_3$$

$$= \frac{1}{2} c_1 e^x - \frac{c}{2c_1} e^{-x} + c_2 x + c_3.$$

105. Integration by Series.—It has already been shown that the general integral of an equation may be found by developing according to Taylor's formula. We shall now give a few examples in which the same result is attained by the method of indeterminate coëfficients.

1. $$ny + \frac{2}{x}\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0.$$

Assume $y = a_1 x^\alpha + a_2 x^\beta + a_3 x^\gamma + a_4 x^\delta +$ etc.,

in which α, β, γ, etc., are ascending powers of x.

Then we shall have

$$ny = na_1 x^\alpha + na_2 x^\beta + na_3 x^\gamma + na_4 x^\delta + \text{etc.};$$

$$\frac{2}{x}\frac{dy}{dx} = 2a_1 \alpha x^{\alpha-2} + 2a_2 \beta x^{\beta-2} + 2a_3 \gamma x^{\gamma-2} + 2a_4 \delta x^{\delta-2} + \text{etc.};$$

$$\frac{d^2y}{dx^2} = a_1 \alpha(\alpha-1) x^{\alpha-2} + a_2 \beta(\beta-1) x^{\beta-2}$$
$$+ a_3 \gamma(\gamma-1) x^{\gamma-2} + a_4 \delta(\delta-1) x^{\delta-2} + \text{etc.}$$

$$\therefore\ ny + \frac{2}{x}\frac{dy}{dx} + \frac{d^2y}{dx^2} = a_1 \alpha(\alpha+1)x^{\alpha-2} + na_1 x^\alpha$$
$$+ a_2 \beta(\beta+1) x^{\beta-2} + na_2 x^\beta + a_3 \gamma(\gamma+1)x^{\gamma-2}$$
$$+ na_3 x^\gamma + a_4 \delta(\delta+1)x^{\delta-2} + na_4 x^\delta + \text{etc.} = 0.$$

This being an identical equation, the coëfficients of the various powers of x must be separately equal to zero.

The lowest power of x being $\alpha - 2$, we must have

$$\alpha(\alpha + 1) = 0, \text{ whence } \alpha = 0, \text{ or } \alpha = -1.$$

1*st.* Let $\alpha = -1$. Now since a_1 is, by supposition, different from zero, the term $na_1 x^{\alpha}$ can not be zero, and its coëfficient na_1 must therefore be reduced with the coëfficients of the next terms.

We must consequently have $\alpha = \beta - 2$ or $\alpha > \beta - 2$.

If $\alpha > \beta - 2$, then we must have the coëfficient of the third term, $\beta(\beta + 1) = 0$, which can only be satisfied by making $\beta = 0$, since β is *greater* than α.

Hence we have $\alpha = -1$; $\beta = 0$; $\gamma - 2 = \alpha$, or $\gamma = 1$;

$$\delta - 2 = \beta, \text{ or } \delta = 2; \text{ etc.};$$

and placing the coëfficients of the different powers respectively equal to zero, we have

$$a_3 \gamma(\gamma + 1) + na_1 = 0; \quad a_4 \delta(\delta + 1) + na_2 = 0; \quad \text{etc.};$$

whence $\quad a_3 = -\dfrac{a_1 n}{1.2}; \qquad a_5 = \dfrac{a_1 n^2}{1.2.3.4};$

$$a_4 = -\frac{a_2 n}{1.2.3}; \quad a_6 = \frac{a_2 n^2}{1.2.3.4.5}, \text{ etc.}$$

Therefore, by substitution in the value of y, we have

$$y = a_1\left(x^{-1} - \frac{nx}{1.2} + \frac{n^2 x^3}{1.2.3.4} \text{ etc.}\right)$$

$$+ a_2\left(x^0 - \frac{nx^2}{1.2.3} + \frac{n^2 x^4}{1.2.3.4.5} \text{ etc.}\right)$$

$$= \frac{a_1}{x} \cos x\sqrt{n} + \frac{a_2}{x} \frac{\sin x\sqrt{n}}{\sqrt{n}}.$$

If we take $\alpha = \beta - 2$, we shall find

$$y = \frac{a_1}{x} \cos x\sqrt{n}, \text{ which is a particular integral.}$$

2*d*. Let $\alpha = 0$. In this case we can not have $\alpha > \beta - 2$, for this would render the coëfficient of $x^{\beta-2}$, or $\beta(\beta+1)$, equal to *zero*,—an impossible condition, since $\beta > \alpha$. We therefore have $\beta - 2 = \alpha$, from which we deduce $\gamma - 2 = \beta$; $\delta - 2 = \gamma$, etc.; and we shall find

$$y = a_1\left(x_0 - \frac{nx^2}{1.\,2.\,3} + \frac{n^2x^4}{1.\,2.\,3.\,4}\text{ etc.}\right) = \frac{a_1}{x}\,\frac{\sin x\sqrt{n}}{\sqrt{n}},$$

a particular integral.

2. $$\frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx} + y = 0.$$

Assuming $y = a_1x^\alpha + a_2x^\beta + a_3x^\gamma + a_4x^\delta + \text{etc.}$, we have

$$\frac{1}{x}\,\frac{dy}{dx} = a_1\alpha x^{\alpha-2} + a_2\beta x^{\beta-2} + a_3\gamma x^{\gamma-2} + a_4\delta x^{\delta-2},\text{ etc.}$$

$$\frac{d^2y}{dx^2} = a_1\alpha(\alpha-1)x^{\alpha-2} + a_2\beta(\beta-1)x^{\beta-2} + a_3\gamma(\gamma-1)x^{\gamma-2},\text{ etc.}$$

The sums of the second members are identically equal to zero, and placing the coëfficient of $x^{\alpha-2}$ equal to zero, we have

$$\alpha + \alpha(\alpha - 1) = 0, \text{ or } \alpha = 0.$$

If we put $\beta - 2 < \alpha$, we shall have, by equating to zero the coëfficient of $x^{\beta-2}$, $\beta = 0$, which is impossible since $\beta > \alpha$. Nor can we put $\beta - 2 > \alpha$, for this would render $a_1 = 0$. Therefore we must have $\beta - 2 = \alpha$, and similarly $\gamma - 2 = \beta$; $\delta - 2 = \gamma$, etc., whence we have

$$\alpha = 0;\quad \beta = 2;\quad \gamma = 4;\quad \delta = 6;\text{ etc.}$$

We shall find

$$a_2 = -\frac{a_1}{2^2};\quad a_3 = \frac{a_1}{2^2.\,4^2};\quad a_4 = \frac{-a_1}{2^2.\,4^2.\,6^2},\text{ etc.; whence}$$

$$y = a_1\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.\,4^2} - \frac{x^6}{2^2.\,4^2.\,6^2},\text{ etc.}\right\},$$

a particular integral, since it involves but one constant.

106. Integration of Linear Equations. — The general form of these equations is

$$\frac{d^m y}{dx^m} + A\frac{d^{m-1}y}{dx^{m-1}} + B\frac{d^{m-2}y}{dx^{m-2}} + \cdots + T\frac{dy}{dx} + Uy = V \quad (1),$$

in which y and its derivatives are of the *first degree*, and $A, B, \ldots T, U, V$ are functions of x.

This equation can not be integrated in its general form, but it possesses several remarkable properties which we proceed to notice.

1*st*. If the last term be wanting, so that

$$\frac{d^m y}{dx^m} + A\frac{d^{m-1}y}{dx^{m-1}} + \cdots + T\frac{dy}{dx} + Uy = 0 \qquad (2);$$

the sum of the m *particular integrals will also be an integral of the equation.*

For if $y', y'', y''' \ldots$ be the particular integrals, we shall have

$$\frac{d^m y'}{dx^m} + \cdots + T\frac{dy'}{dx} + Uy' = 0;$$

$$\frac{d^m y''}{dx^m} + \cdots + T\frac{dy''}{dx} + Uy'' = 0;$$

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Adding these equations we shall have the same result as if we had substituted $y' + y'' + y''' +$ etc. for y in (2).

Therefore, if y', y'', etc. are the m particular integrals, their sum $y' + y'' + y''' +$ etc. will also be an integral.

Moreover, the sum of any number of particular integrals will be a particular integral; and since the product of any particular integral by a constant factor will evidently be a particular integral, it follows that

$$c'\, y' + c''\, y'' + \ldots c^m\, y^m,$$

in which $c', c'' \dots c^m$ are arbitrary constants, will be the *general* integral; so that we shall have

$$y = c'\, y' + c''\, y'' + c'''\, y''' + \dots + c^m\, y^m.$$

2*d*. Let the last equation be the general integral of (2). Then it is obvious that the general integral of (1) may be obtained from this by replacing the arbitrary constants by functions of x.

We shall now show how these functions of x may be determined, and as the method of demonstration would be the same for equations of every order, we shall, for the sake of simplicity, suppose the equation to be of the third order.

Differentiating the equation

$$y = c'\, y' + c''\, y'' + c'''\, y''', \text{ we have}$$

$$dy = c'\, dy' + c''\, dy'' + c'''\, dy''' + y'\, dc' + y''\, dc'' + y'''\, dc'''.$$

Assuming, for the first condition by which c', c'', c''' are to be determined, the equation

$$y'\, dc' + y''\, dc'' + y'''\, dc''' = 0, \text{ we have}$$

$$dy = c'\, dy' + c''\, dy'' + c'''\, dy''' \qquad (a).$$

Differentiating again, we have

$$d^2y = c'\, d^2y' + c''\, d^2y'' + c'''\, d^2y''' + dy'\, dc' + dy''\, dc'' + dy'''\, dc'''.$$

Assuming, for the second condition between c', c'', c''',

$$dy'\, dc' + dy''\, dc'' + dy'''\, dc''' = 0, \text{ we have}$$

$$d^2 y = c'\, d^2y' + c''\, d^2y'' + c'''\, d^2y''' \qquad (b).$$

Differentiating again, we have

$$d^3y = c'\, d^3y' + c''\, d^3y'' + c'''\, d^3y''' + d^2y'\, dc' + d^2y''\, dc'' + d^2y'''\, dc''' \qquad (c).$$

Assuming, for the third condition between c', c'', c''',

$$d^2y'\,dc' + d^2y''\,dc'' + d^2y'''\,dc''' = Vdx^3,$$

we shall have the three equations of condition:

$$y'\,dc' + y''\,dc'' + y'''\,dc''' = 0,$$

$$dy'\,dc' + dy''\,dc'' + dy'''\,dc''' = 0,$$

$$d^2y'\,dc' + d^2y''\,dc'' + d^2y'''\,dc''' = Vdx^3.$$

From these three equations the values of dc', dc'', dc''', and consequently those of c', c'', c''', can be determined in functions of x by elimination and integration; and if the values of dy, d^2y, d^3y in equations (a), (b), and (c) be substituted in the equation

$$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Uy = V,$$

the resulting equation will be identical.

Whence it follows, in general, that if we can find

$$y = c'y' + c''y'' + \ldots + c^m y^m,$$

the general integral of

$$\frac{d^my}{dx^m} + \cdots + Uy = 0,$$

that of $$\frac{d^my}{dx^m} + \cdots + Uy = V$$

can be found by replacing c', c'', etc., by functions of x, determined as above.

107. Linear equations of the particular form

$$\frac{d^my}{dx^m} + \frac{A}{ax+b}\frac{d^{m-1}y}{dx^{m-1}} + \frac{B}{(ax+b)^2}\frac{d^{m-2}y}{dx^{m-2}} + \cdots + \frac{U}{(ax+b)^m} = 0$$

may be readily integrated by assuming $y = (ax + b)^a$.

For, differentiating this expression and substituting in the given equation, we have

$$a(a-1)\dots(a-m+1)a^m+Aa(a-1)\\ \dots(a-m+2)a^{m-1}+\dots+U=0.$$

Let $a', a'', a''' \dots a^m$ be the values of a derived from this equation. Then the particular integrals will be

$$y'=(ax+b)^{a'};\ y''=(ax+b)^{a''};\ \dots\ y^m=(ax+b)^{a^m};$$

and the general integral is

$$y=c'(ax+b)^{a'}+c''(ax+b)^{a''}+\dots+c^m(ax+b)^{a^m}.$$

108. Linear equations in which the coëfficients are all *constants*, are also readily integrable.

Let us take the equation

$$\frac{d^m y}{dx^m}+A\frac{d^{m-1}y}{dx^{m-1}}+\dots+T\frac{dy}{dx}+Uy=V \quad (1).$$

If we assume $y=y+\frac{V}{U}$, the last term will disappear, and we may therefore suppose the second member to be *zero*.

Let $y=e^{ax}$. Then, by differentiation and substitution, we shall have

$$a^m+Aa^{m-1}+\dots Ta+U=0 \quad (2).$$

If, now, we can find the m roots of this equation,

$$a',\ a'',\ a''' \dots a^m,$$

it is evident that we shall have for the m particular integrals the following values:

$$y'=e^{a'x};\ y''=e^{a''x};\ \dots\ y^m=e^{a^m x};$$

and the general integral will be

$$y=c'e^{a'x}+c''e^{a''x}+\dots c^m e^{a^m x} \quad (3).$$

109. If the equation (2) has *imaginary* roots, the corresponding terms of the general integral will be imaginary also, but they may easily be rendered real.

Let the imaginary roots be of the form $\alpha \pm \beta \sqrt{-1}$.

Then the corresponding terms of the general integral will be of the forms

$$Ae^{(\alpha+\beta\sqrt{-1})x} + Be^{(\alpha-\beta\sqrt{-1})x}; \text{ or}$$

$$e^{\alpha x}(Ae^{\beta\sqrt{-1}\,x} + Be^{-\beta\sqrt{-1}\,x}); \text{ or}$$

$$e^{\alpha x}A\{\cos\beta x + \sqrt{-1}\sin\beta x\} + e^{\alpha x}B\{\cos\beta x - \sqrt{-1}\sin\beta x\}.$$

A and B being arbitrary, we may determine them by the conditions

$$A + B = M; \quad (A - B)\sqrt{-1} = N;$$

whence we have for the values of the imaginary terms

$$Me^{\alpha x}\cos\beta x + Ne^{\alpha x}\sin\beta x,$$

and similar forms may be found for any two conjugate imaginary terms in the general integral.

110. If (2) has *equal* roots, the value of y in (3) will not be the general integral, since it will not then contain m constants. We may, however, find the general integral in this case as follows:

Suppose that the equation has *two* equal roots. If we change the coëfficients by infinitesimal amounts, (2) will no longer have equal roots, and (3) will be the general integral. The *limit* to this general integral will evidently be the general integral of the given equation.

Let a' and a'' be the two equal roots. If for a'' we write $a' + \delta$, we shall evidently have for the general integral of the new equation

$$y = c'e^{a'x} + c''e^{(a'+\delta)x} + \text{etc.}$$

$$= c'e^{a'x} + c''e^{a'x}\left(1 + \delta x + \frac{\delta^2 x^2}{1.2} + \text{etc.}\right) + \text{etc.}$$

Making $c' + c'' = A$, and $c''\delta = B$, we have

$$y = e^{a'x}\{A + Bx\} + \left\{\frac{\delta B}{1.2} x^2 e^{a'x} + \cdots \right\} + \text{etc.};$$

and, passing to the limits,

$$y = e^{a'x}\{A + Bx\} + c''' e^{a'''x} + \ldots \text{ etc.},$$

which is the general integral of the given equation. Similarly, if there were three equal roots, we should find

$$y = e^{a'x}\{A + Bx + Cx^2\} + c^{\text{iv}} e^{a^{\text{iv}}x} + \ldots \text{ etc.};$$

and so on for any number of equal roots.

111. Examples of Linear Equations.

1. $$\frac{d^2u}{d\theta^2} + b^2 u = 0.$$

Let $u = e^{a\theta}$. Then $\dfrac{du}{d\theta} = ae^{a\theta}$, and $\dfrac{d^2u}{d\theta^2} = a^2 e^{a\theta}$.

$$\therefore\ a^2 e^{a\theta} + b^2 e^{a\theta} = 0, \text{ and } a^2 + b^2 = 0.$$

$\therefore\ a = \pm b\sqrt{-1}$, and

$$u = c' e^{b\theta\sqrt{-1}} + c'' e^{-b\theta\sqrt{-1}}$$

$$= (c' + c'') \cos b\theta + (c' - c'')\sqrt{-1} \sin b\theta$$

$$= M \cos b\theta + N \sin b\theta.$$

2. $$\frac{d^2u}{d\theta^2} + a^2 u + b^2 = 0.$$

Putting $b^2 = a^2 k$, and $u + k = \omega$, we have

$$\frac{d^2\omega}{d\theta^2} + a^2 \omega = 0,$$

which is integrable by the last example.

3. $$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$$

Putting $y = e^{ax}$, we have

$$a^3 - 6a^2 + 11a - 6 = 0,$$

the roots of which equation are 1, 2, 3. Therefore the general integral is

$$y = c' e^x + c'' e^{2x} + c''' e^{3x}.$$

4. $$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$$

Putting $y = e^{ax}$, we have

$$a^2 + 8a + 16 = 0.$$

This equation has two roots equal to -4. The general integral is, therefore,

$$y = e^{-4x}\{A + Bx\}.$$

5. $$\frac{d^2s}{dt^2} + 2k\frac{ds}{dt} + fs = 0.$$

Putting $s = e^{mt}$, we have

$$m^2 + 2km + f = 0,$$

the roots of which are $-k \pm \sqrt{-1}\sqrt{f-k^2}$.

Therefore the general integral is

$$s = e^{-kt}\{c' e^{at\sqrt{-1}} + c'' e^{-at\sqrt{-1}}\} = e^{-kt}\{M\cos at + N\sin at\},$$

in which $a = \sqrt{f-k^2}$.

6. $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x.$$

The solution of the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0, \text{ is}$$

$$y = c' e^{3x} + c'' e^{2x}$$

Differentiating, we have

$$\frac{dy}{dx} = 3c' e^{3x} + 2c'' e^{2x} + e^{3x}\frac{dc'}{dx} + e^{2x}\frac{dc''}{dx};$$

$$\therefore \quad e^{3x}\frac{dc'}{dx} + e^{2x}\frac{dc''}{dx} = 0 \text{ [Art. 106]}.$$

$$\frac{d^2y}{dx^2} = 9c' e^{3x} + 4c'' e^{2x} + 3e^{3x}\frac{dc'}{dx} + 2e^{2x}\frac{dc''}{dx};$$

$$\therefore \quad 3e^{3x}\frac{dc'}{dx} + 2e^{2x}\frac{dc''}{dx} = x.$$

From these two equations of condition, we have

$$dc' = e^{-3x}x\,dx, \qquad \therefore \quad c' = C' - \frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}.$$

$$dc'' = -e^{-2x}x\,dx, \quad \therefore \quad c'' = C'' + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}.$$

$$\therefore \quad y = C' e^{3x} + C'' e^{2x} - \frac{1}{3}x - \frac{1}{9} + \frac{x}{2} + \frac{1}{4}$$

$$= C' e^{3x} + C'' e^{2x} + \frac{1}{6}\left(x + \frac{5}{6}\right)$$

7. $$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$$

Ans. $y = e^x\{A + Bx + Cx^2\}$.

CHAPTER XVI.

INTEGRATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

112. It is a problem of frequent occurrence in the applications of the Calculus, to determine a finite equation between two or more variables from the data furnished by a number of differential equations between the same variables.

We shall in the present chapter offer a few examples in which this operation can be effected.

113. Let there be the two equations,

$$\frac{dx}{dt} + ay = 0; \quad \frac{dy}{dt} + bx = 0.$$

Differentiating the first equation with reference to t, we have

$$\frac{d^2x}{dt^2} + a\frac{dy}{dt} = 0,$$

and the elimination of $\frac{dy}{dt}$ between this and the second equation gives

$$\frac{d^2x}{dt^2} - abx = 0.$$

The integral of this equation is

$$x = c' e^{mt} + c'' e^{-mt} \qquad (1),$$

in which $m = \sqrt{ab}$.

Differentiating (1), and substituting the value of $\frac{dx}{dt}$ in the first equation, we have

$$y = c''\sqrt{\frac{b}{a}}\, e^{-mt} - c'\sqrt{\frac{b}{a}}\, e^{mt} \qquad (2).$$

The elimination of mt between (1) and (2) will give a finite equation between x and y.

114. Let the two equations be

$$\frac{dx}{dt} + ax + by = 0; \quad \frac{dy}{dt} + a'x + b'y = 0.$$

If we multiply the second by θ and add the result to the first, we shall have

$$\frac{d(x+\theta y)}{dt} + (a + a'\theta)x + (b + b'\theta)y = 0.$$

Putting $x + \theta y = v$; $a + a'\theta = -\alpha$; $b + b'\theta = -\alpha\theta$, this equation becomes

$$\frac{dv}{dt} - \alpha v = 0; \text{ whence } v = ce^{\alpha t}.$$

Now determining θ from the two conditions

$$a + a'\theta = -\alpha; \quad b + b'\theta = -\alpha\theta,$$

let its two resulting values be denoted by θ' and θ'', and let the corresponding values of α be α' and α''.

Then we shall have

$$x + \theta' y = c' e^{\alpha' t},$$

$$x + \theta'' y = c'' e^{\alpha'' t},$$

from which the relation between x and y can be found by the elimination of t.

115. Let the equations be

$$\frac{dx}{dt} + ax + by = T \qquad (1),$$

$$\frac{dy}{dt} + a'x + b'y = T' \qquad (2),$$

in which T and T' are functions of t.

Multiplying (2) by θ, and adding the result to (1), we have

$$\frac{d(x+\theta y)}{dt} + (a + a'\theta)x + (b + b'\theta)y = T + T'\theta \qquad (3).$$

Assuming $a + a'\theta = -\alpha$; $b + b'\theta = -\alpha\theta$; $x + \theta y = v$, (3) becomes

$$\frac{dv}{dt} - \alpha v = T + T'\theta.$$

This is a linear equation of the first order, and its integral is

$$v = e^{\alpha t}\left\{c + \int (T + T'\theta)\, e^{-\alpha t} dt\right\}.$$

If we determine the values of α and θ as in the last example, we shall have the two equations

$$x + \theta' y = e^{\alpha' t}\left\{c + \int (T + T'\theta')\, e^{-\alpha' t}\, dt\right\},$$

$$x + \theta'' y = e^{\alpha'' t}\left\{c' + \int (T + T'\theta'') e^{-\alpha'' t}\, dt\right\},$$

from which the values of x and y may be determined in terms of t.

EXAMPLE. $$\frac{dx}{dt} + 4x + 3y = t.$$

$$\frac{dy}{dt} + 2x + 5y = e^t.$$

We have

$$\frac{d(x + \theta y)}{dt} + (4 + 2\theta)x + (3 + 5\theta)\, y = t + \theta e^t.$$

Now, assuming $4 + 2\theta = -\alpha$;

$3 + 5\theta = -\alpha\theta = 4\theta + 2\theta^2$, we have

$$2\theta^2 - \theta = 3; \text{ whence } \theta' = -1,\ \theta'' = \frac{3}{2}.$$

$$\therefore\ 4 - 2 = -\alpha' = 2, \text{ and } 4 + 3 = -\alpha'' = 7.$$

$$\therefore\ x - y = e^{-2t}\left\{c + \int (t - e^t)\, e^{2t}\, dt\right\},$$

$$x + \frac{3}{2}\, y = e^{-7t}\left\{c' + \int (t + \frac{3}{2}\, e^t)\, e^{7t}\, dt.\right.$$

These equations are readily integrable.

116. Let us now take the three equations

$$\frac{dx}{dt} + ax + by + cz = T \qquad (1),$$

$$\frac{dy}{dt} + a'x + b'y + c'z = T' \qquad (2),$$

$$\frac{dz}{dt} + a''x + b''y + c''z = T'' \qquad (3),$$

in which T, T', T'' are functions of t.

Multiply (2) by θ, and (3) by θ_1, and add the results to (1). We thus obtain

$$\frac{d(x + \theta y + \theta_1 z)}{dt} + (a + a'\theta + a''\theta_1)x + (b + b'\theta + b''\theta_1)y$$
$$+ (c + c'\theta + c''\theta_1)z = T + T'\theta + T''\theta_1.$$

Putting $x + \theta y + \theta_1 z = v$; $\quad a + a'\theta + a''\theta_1 = -\alpha$;

$$b + b'\theta + b''\theta_1 = -\alpha\theta; \quad c + c'\theta + c''\theta_1 = -\alpha\theta_1,$$

this equation becomes

$$\frac{dv}{dt} - \alpha v = T + T'\theta + T''\theta_1,$$

which is linear, and may be integrated as in the preceding examples.

Now, from the three equations of condition between θ and θ_1, there will result two cubic equations from which we can find three values each for θ and θ_1. Calling these values θ', θ'', θ''', θ_1', θ_1'', θ_1''', and designating the corresponding values of v by v', v'', v''', we shall have the three equations

$$x + \theta' y + \theta_1' z = v',$$

$$x + \theta'' y + \theta_1'' z = v'',$$

$$x + \theta''' y + \theta_1''' z = v''',$$

from which x, y and z can be determined in terms of t.

117. Let the given equations be the following of the second order:

$$\frac{d^2 x}{dt^2} + ax + by + c = 0 \qquad (1),$$

$$\frac{d^2 y}{dt^2} + a'x + b'y + c' = 0 \qquad (2).$$

Multiplying (2) by θ and adding to (1), the resulting equation may be written

$$\frac{d^2}{dt^2}\left\{x + \theta y + \frac{c + c'\theta}{a + a'\theta}\right\}$$
$$+ (a + a'\theta)\left\{x + \left(\frac{b + b'\theta}{a + a'\theta}\right)y + \frac{c + c'\theta}{a + a'\theta}\right\} = 0.$$

Making $x + \theta y + \dfrac{c + c'\theta}{a + a'\theta} = v$; $\dfrac{b + b'\theta}{a + a'\theta} = \theta$;

$a + a'\theta = - n^2$, this equation becomes

$$\frac{d^2 v}{dt^2} - n^2 v = 0,$$

the integral of which is

$$v = Ce^{nt} + C'e^{-nt}.$$

Designating by θ', θ'' the values of θ derived from the equation $\dfrac{b + b'\theta}{a + a'\theta} = \theta$, we shall have

$$x + \theta' y + \frac{c + c'\theta'}{a + a'\theta'} = Ce^{n't} + C'e^{-n't},$$

$$x + \theta'' y + \frac{c + c'\theta''}{a + a'\theta''} = C_1 e^{n''t} + C_1' e^{-n''t}.$$

Example.

$$\frac{d^2 x}{dt^2} - 3x - 4y + 3 = 0;$$

$$\frac{d^2 y}{dt^2} + x - 8y + 5 = 0.$$

We have $a = -3$; $a' = 1$; $b = -4$; $b' = -8$; $c = +3$; $c' = 5$; and we shall find

$$\theta' = -4; \quad \theta'' = -1; \quad n' = \sqrt{7}; \quad n'' = 2;$$

$$\frac{c + c'\theta'}{a + a'\theta'} = \frac{17}{7}; \qquad \frac{c + c'\theta''}{a + a'\theta''} = \frac{1}{2}.$$

Hence, $\quad x - 4y + \dfrac{17}{7} = Ce^{t\sqrt{7}} + C'e^{-t\sqrt{7}},$

$$x - y + \frac{1}{2} = C_1 e^{2t} + C_1' e^{-2t},$$

from which we may find an algebraic equation between x and y by the elimination of t.

118. As a last example let us take the two equations

$$\frac{d^2 x}{dt^2} + \frac{mx}{r^3} = 0 \qquad (1),$$

$$\frac{d^2 y}{dt^2} + \frac{my}{r^3} = 0 \qquad (2),$$

in which $r = \sqrt{x^2 + y^2}$.

If we multiply the first equation by y, the second by x, and subtract, we shall have

$$x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = 0,$$

the integral of which is

$$x \frac{dy}{dt} - y \frac{dx}{dt} = c \qquad (3).$$

Transposing the first term of (1), and multiplying by (3), we have

$$-c \frac{d^2 x}{dt^2} = \frac{m}{r^3}\left\{x^2 \frac{dy}{dt} - xy \frac{dx}{dt}\right\} = m \frac{d}{dt}\left(\frac{y}{r}\right),$$

the integral of which is

$$-c \frac{dx}{dt} = m \frac{y}{r} + a \qquad (4).$$

Similarly, we shall obtain from (2) the equation

$$c\,\frac{dy}{dt} = m\,\frac{x}{r} + b \qquad (5).$$

Multiplying (4) and (5) by y and x, respectively, and adding the results, we find

$$c\left\{x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right\} = m\,\frac{(x^2+y^2)}{r} + ay + bx \qquad (6),$$

$$\text{or} \quad mr + ay + bx = c^2 \qquad (7),$$

$$\text{or} \quad m^2(x^2+y^2) = \{c^2 - (ay+bx)\}^2 \qquad (8).$$

This is the partial solution of the problem "to find the motion of a particle when attracted to a fixed point by a force which varies inversely as the square of the distance;" and equation (8), being of the second degree, shows that the path of such a particle will be a Conic Section.